PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Numerical Results of the Theory of the Diffraction of a Plane Electromagnetic Wave by a Perfectly Conducting Sphere

J. Proudman, A. T. Doodson and G. Kennedy

Phil. Trans. R. Soc. Lond. A 1918 **217**, 279-314
doi: 10.1098/rsta.1918.0008

VIII. *Numerical Results of the Theory of the Diffraction of a Plane Electromagnetic Wave by a Perfectly Conducting Sphere.*

By J. PROUDMAN, A. T. DOODSON, *and* G. KENNEDY.

Communicated by T. J. I'A. BROMWICH, *Sc.D., F.R.S.*

Received April 13,—Read November 23, 1916.

Introduction.

(*By* J. PROUDMAN.)

1. AT the suggestion of Dr. BROMWICH, I began the computations leading to this paper nearly three years ago. Using tables constructed by Lord RAYLEIGH* and Prof. A. LODGE,* I obtained results for $\kappa a = 1, 2, 10$† and $\theta = 0°, 180°; 90°; 45°, 135°; 20°, 160°; 70°, 110°$; in this order. From the results for $\kappa a = 1$ and 2, graphs of Y_1, Y_2, Z_1, Z_2 could be constructed with some confidence, but such graphs were entirely impossible in the case of $\kappa a = 10$, owing to the large number of their undulations. (For the graphs of these functions, as finally drawn, see figs. 1, 3, 18, 20, 22, 24.)

I then handed over the work to Messrs. DOODSON and KENNEDY, and the whole of the results as they now appear are due to them. Mr. DOODSON first constructed tables‡ for BESSEL'S functions of half-integral orders, and Mr. KENNEDY constructed tables§ for the derivatives of LEGENDRE'S functions. These two sets of tables, together with those of LODGE already quoted, are what have been used in all the subsequent work.

Mr. DOODSON computed quite independently the cases of $\kappa a = 1, 2$, for all the values of θ that I had taken together with $\theta = 10°, 170°; 30°, 150°; 60°, 120°; 80°, 100°$. His results were in agreement with mine, except for a number of small differences

* RAYLEIGH, "On the Acoustic Shadow of a Sphere, with an Appendix giving the Values of LEGENDRE'S Functions, . . ., by Prof. A. LODGE," 'Phil. Trans. Roy. Soc.,' A, vol. cciii., p. 87 (1904); ['Sc. Papers,' vol. v., p. 149].

RAYLEIGH, "Incidence of Light upon a Transparent Sphere of Dimensions comparable with a Wave-length," 'Roy. Soc. Proc.,' A, vol. lxxxiv., p. 25 (1910); ['Sc. Papers,' vol. v., p. 547].

† The notation is explained in the next section.

‡ These have been published by the British Association; 'Report' for 1914, p. 87.

§ These are to be presented shortly to the British Association.

 [Published August 15, 1917.

which he found to be due to numerical errors on my part. From the larger number of values of θ thus taken, graphs of $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$ (figs. 2, 4) could be constructed with confidence and no further work was necessary for $\kappa a = 1, 2$. Tables X., XI. ($\kappa a = 1, 2$), and figs. 1 to 4, contain the results of this part of Mr. DOODSON'S work.

Sharing equally the labour, Messrs. DOODSON and KENNEDY then similarly recomputed the case of $\kappa a = 10$, and added the extra values of θ which had been taken for $\kappa a = 1, 2$. It then proved possible to construct rough graphs of Y_1, Y_2, Z_1, Z_2, but the uncertainties in the graphs of $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$ were very great, owing to the magnification of errors involved in the squaring. In order to get more insight into the general nature of the results, they performed for $\kappa a = 9$ all the work that they had done for $\kappa a = 10$, sharing it similarly. The construction of accurate graphs of $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$ proved as impossible as before, *i.e.*, knowing only the points on the curves of figs. 25 to 28, indicated by small circles, the complete curves could not be drawn.

Mr. DOODSON then conceived the idea of calculating the gradients of the curves at all the values of θ taken, and formed the necessary series. Mr. KENNEDY shared the computations with him, and the gradients were calculated at every known point of the curves for both $\kappa a = 9$ and 10. These proved a great help.

At this stage the results obtained for $\kappa a = 9, 10$ were those indicated by the small circles and tangents in figs. 17 to 28. The small uncertainties in the graphs of Y_1, Y_2, Z_1, Z_2, still led to considerable uncertainties in those of $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$. Mr. DOODSON then designed and carried out an examination of the way in which the results found differ from those given by the theoretical first approximation for large values of κa. The sections on analysis of results and interpolation give this work. Some of the methods proved their utility by leading to the detection of certain errors in the previous work, and all of them were used for interpolation purposes in constructing the final curves.

Only the curves, as finally constructed from all considerations, are now given.

In the work of drawing up the results, all the curves have been drawn by Mr. DOODSON, while most of the tables have been written out by Mr. KENNEDY. Only a selection, however, is printed; the remainder are in the possession of the Royal Society. For instance, Tables I. to VIII. are only printed in so far as they refer to $\kappa a = 10$, while Tables XII. to XIX., which refer to the gradients, are omitted altogether.

General Formulæ.

2. For the theory of the problem we shall quote a paper by Dr. BROMWICH on "The Scattering of Plane Waves by Spheres."*

* This is a paper which Dr. BROMWICH communicated to the Society at the same time as the present one.

Using spherical polar co-ordinates r, θ, ϕ, the incident plane wave train is taken to contain the time factor $e^{i\kappa ct}$, to travel along the negative direction of the axis $\theta = 0$, to be polarized in the plane $\phi = \frac{1}{2}\pi$, and to have the electric force of unit amplitude. The components of electric and magnetic force, in the disturbance produced by the sphere, along the directions of r, θ, ϕ increasing, are denoted respectively by X, Y, Z and α, β, γ. Then at a distance from the sphere, which is large compared with a wave-length of the incident train, we have, from the paper quoted,

$$\left.\begin{aligned} \mathrm{X} &= \quad c\alpha = 0, \\ \mathrm{Y} &= \quad c\gamma = \frac{\partial \mathrm{M}}{\partial \theta} - \frac{\partial \mathrm{N}}{\sin\theta \partial \phi}, \\ \mathrm{Z} &= -c\beta = \frac{\partial \mathrm{M}}{\sin\theta \partial \phi} + \frac{\partial \mathrm{N}}{\partial \theta}, \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where

$$\left.\begin{aligned} \mathrm{M} &= \cos\phi \frac{e^{-i\kappa(r-ct)}}{\kappa r} \sin\theta \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \frac{\mathrm{S}'_n(\kappa a)}{\mathrm{E}'_n(\kappa a)} \mathrm{P}'_n(\cos\theta), \\ \mathrm{N} &= \sin\phi \frac{e^{-i\kappa(r-ct)}}{\kappa r} \sin\theta \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \frac{\mathrm{S}_n(\kappa a)}{\mathrm{E}_n(\kappa a)} \mathrm{P}'_n(\cos\theta). \end{aligned}\right\} \quad . \quad . \quad (2)$$

Take now

$$\left.\begin{aligned} \mathrm{Y} &= \cos\phi \frac{e^{-i\kappa(r-ct)}}{\kappa r} (\mathrm{Y}_1 + i\mathrm{Y}_2), \\ \mathrm{Z} &= \sin\phi \frac{e^{-i\kappa(r-ct)}}{\kappa r} (\mathrm{Z}_1 + i\mathrm{Z}_2), \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

where Y_1, Y_2, Z_1, Z_2 are real. Then the time-means of the squares of the real parts of Y and Z are respectively

$$\tfrac{1}{2} \frac{\cos^2\phi}{(\kappa r)^2} (\mathrm{Y}_1^2 + \mathrm{Y}_2^2), \qquad \tfrac{1}{2} \frac{\sin^2\phi}{(\kappa r)^2} (\mathrm{Z}_1^2 + \mathrm{Z}_2^2). \quad . \quad . \quad . \quad (4)$$

These give a measure of the energy of the disturbance.

The functions Y_1, Y_2, Z_1, Z_2 involve only κa and θ. It is the object of the present paper to evaluate them, together with $\mathrm{Y}_1^2+\mathrm{Y}_2^2$ and $\mathrm{Z}_1^2+\mathrm{Z}_2^2$ for certain values of κa, and all values of θ from 0 to π.

Writing $\cos\theta = \mu$, we have

$$\frac{d}{d\theta}\{\sin\theta \mathrm{P}'_n(\mu)\} = n(n+1)\mathrm{P}_n(\mu) - \mu \mathrm{P}'_n(\mu),$$

using which we obtain

$$\left.\begin{aligned} \mathrm{Y}_1 &= \mathrm{A} + \mu \mathrm{A}' - \mathrm{C}', \qquad & \mathrm{Y}_2 &= \mathrm{B} + \mu \mathrm{B}' - \mathrm{D}', \\ \mathrm{Z}_1 &= \mathrm{A}' + \mu \mathrm{A} - \mathrm{C}, \qquad & \mathrm{Z}_2 &= \mathrm{B}' + \mu \mathrm{B} - \mathrm{D}, \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

where

$$\left.\begin{aligned}
A &= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \frac{S_n(\kappa a)\, C_n(\kappa a)}{|E_n(\kappa a)|^2} P'_n(\mu), \\
B &= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \frac{S_n^2(\kappa a)}{|E_n(\kappa a)|^2} P'_n(\mu), \\
C &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S_n(\kappa a)\, C_n(\kappa a)}{|E_n(\kappa a)|^2} P_n(\mu), \\
D &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S_n^2(\kappa a)}{|E_n(\kappa a)|^2} P_n(\mu),
\end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

and

$$\left.\begin{aligned}
A' &= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \frac{S'_n(\kappa a)\, C'_n(\kappa a)}{|E'_n(\kappa a)|^2} P'_n(\mu), \\
B' &= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \frac{S'^2_n(\kappa a)}{|E'_n(\kappa a)|^2} P'_n(\mu), \\
C' &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S'_n(\kappa a)\, C'_n(\kappa a)}{|E'_n(\kappa a)|^2} P_n(\mu), \\
D' &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S'^2_n(\kappa a)}{|E'_n(\kappa a)|^2} P_n(\mu).
\end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

Further, we easily obtain

$$\left.\begin{aligned}
\frac{\partial Y_1}{\partial \mu} &= \frac{\mu Y_1 + Z_1}{1-\mu^2} - \frac{\partial C'}{\partial \mu}, \qquad & \frac{\partial Y_2}{\partial \mu} &= \frac{\mu Y_2 + Z_2}{1-\mu^2} - \frac{\partial D'}{\partial \mu}, \\
\frac{\partial Z_1}{\partial \mu} &= \frac{\mu Z_1 + Y_1}{1-\mu^2} - \frac{\partial C}{\partial \mu}, \qquad & \frac{\partial Z_2}{\partial \mu} &= \frac{\mu Z_2 + Y_2}{1-\mu^2} - \frac{\partial D}{\partial \mu},
\end{aligned}\right\} \quad \ldots \ldots \quad (8)$$

while if we denote $\partial C/\partial\mu$, $\partial D/\partial\mu$, $\partial C'/\partial\mu$, $\partial D'/\partial\mu$ by c, d, c', d' respectively, we have

$$\left.\begin{aligned}
c &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S_n(\kappa a)\, C_n(\kappa a)}{|E_n(\kappa a)|^2} P'_n(\mu), \\
d &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S_n^2(\kappa a)}{|E_n(\kappa a)|^2} P'_n(\mu), \\
c' &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S'_n(\kappa a)\, C_n'(\kappa a)}{|E'_n(\kappa a)|^2} P'_n(\mu), \\
d' &= \sum_{n=1}^{\infty} (-1)^n (2n+1) \frac{S'^2_n(\kappa a)}{|E'_n(\kappa a)|^2} P'_n(\mu).
\end{aligned}\right\} \quad \ldots \ldots \quad (9)$$

In summing the series in (6), (7), (9), supplementing values of θ may be considered at the same time if the odd and even terms of these series are taken separately. We shall use the suffixes 1, 2, to denote respectively the sums of odd terms and sums of even terms. Then for supplementary values of θ, the two values of each of

$$A_1,\ B_1,\ A'_1,\ B'_1,\ C_2,\ D_2,\ C'_2,\ D'_2,\ c_1,\ d_1,\ c'_1,\ d'_1,$$

are equal, while the two values of each of

$$A_2,\ B_2,\ A'_2,\ B'_2,\ C_1,\ D_1,\ C'_1,\ D'_1,\ c_2,\ d_2,\ c'_2,\ d'_2,$$

are equal and opposite. For this reason in computing Y_1, for example, we evaluate

$$A_1+\mu A'_2-C'_2, \quad A_2+\mu A'_1-C'_1,$$

separately for $0 \leq \theta \leq \frac{1}{2}\pi$.

For $\theta = 0$ we have

$$2P'_n(\mu) = n(n+1)P_n(\mu),$$

so that

$$C_1 = 2A_1, \quad D_1 = 2B_1, \quad C'_1 = 2A'_1, \quad D'_1 = 2B'_1,$$

$$C_2 = 2A_2, \quad D_2 = 2B_2, \quad C'_2 = 2A'_2, \quad D'_2 = 2B'_2.$$

Numerical Summation of Series.

(*By* A. T. Doodson *and* G. Kennedy.)

3. The first step was to construct tables of

$$\log(2n+1)F_n(\kappa a) \quad \text{and} \quad \log\frac{2n+1}{n(n+1)}F_n(\kappa a),$$

where $F_n(\kappa a)$ takes each of the forms

$$\frac{|S_n(\kappa a)\,C_n(\kappa a)|}{|E_n(\kappa a)|^2}, \qquad \frac{S_n^2(\kappa a)}{|E_n(\kappa a)|^2}, \qquad \frac{|S'_n(\kappa a)\,C'_n(\kappa a)|}{|E'_n(\kappa a)|^2}, \qquad \frac{S'^2_n(\kappa a)}{|E'_n(\kappa a)|^2}.$$

This was done by means of the Brit. Assoc. tables* mentioned in § 1, using seven-figure logarithms and afterwards reducing to five figures. To each of these was added $\log|P_n(\mu)|$ or $\log|P'_n(\mu)|$ for the same value of n, using five-figure logarithms.

* In these tables the values of $|E_n(9)|^2$ for $n = 13, 14$ are misprinted; the decimal point requires moving one place to the left in each case. Care should be taken in using the logarithms, as some of the negative characteristics are printed without the sign placed over them.

Checks were devised to secure accurate addition, and no errors were afterwards discovered in this part of the work.

Tables* I. to VIII. were then constructed by taking anti-logarithms. This was a step the accuracy of which was not easily checked. The method used was to repeat the work with a different book of logarithms and under different conditions so as to avoid "repetition" errors. Only one or two errors were afterwards found in this part of the work. More errors were made in the summation of odd terms and of even terms of the series, the results of which are given at the feet of Tables I. to VIII. Only five or six, however, which escaped immediate detection by checking, were afterwards found in Table X. How these were detected will be described in the section on analysis of results. Except in the cases of $\kappa a = 1, 2$, one person computed A, B, C, D and the other A′, B′, C′, D′. The final work of combining these functions was done separately by each and the results compared.

It has been the practice throughout the work to use more figures than were strictly necessary for the desired degree of accuracy. For example, many terms in the anti-logarithms are given to six figures, though only five-figure logarithms were used. This was found conducive to speed and accuracy, but it is not intended that the tables should be regarded as accurate to the extent given, except when this is stated.

The final results for Y_1, Y_2, Z_1, Z_2 are probably accurate to at least three decimal places, while $Y_1^2+Y_2^2$ and $Z_1^2+Z_2^2$ can safely be given to four significant figures, as is done in the tables.

The derivatives c', d', c, d were computed similarly. The gradients are probably accurate to the order given in the tables.†

* The reason that $\theta = 45°$ has been taken where a regular sequence would require $\theta = 40°, 50°$, appears from the account of the history of the work in § 1.

† As stated at the end of § 1, these are not printed. The unprinted tables referring to this section are as follows:—

Table	IX.	giving	$A_1+\mu A'_2-C_2$, $A_2+\mu A'_1-C'_1$, &c.
,,	XII.	,,	terms of the series c.
,,	XIII.	,,	,, ,, d.
,,	XIV.	,,	,, ,, c'.
,,	XV.	,,	,, ,, d'.
,,	XVI.	,,	$\partial C'/\partial\mu$, $\partial D'/\partial\mu$, &c.
,,	XVII.	,,	$\partial Y_1/\partial\mu$, $\partial Y_2/\partial\mu$, &c.
,,	XVIII.	,,	$\partial Y_1/\partial\theta$, $\partial Y_2/\partial\theta$, &c.
,,	XIX.	,,	$\partial(Y_1^2+Y_2^2)/\partial\theta$, $\partial(Z_1^2+Z_2^2)/\partial\theta$.

Table I.

$$(-1)^n \cdot \frac{2n+1}{n\,(n+1)} \cdot \frac{S_n C_n}{|E_n|^2} \cdot P'_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1	+0·73176	+0·73176	+0·73176	+0·73176	+0·73176	+0·73176	+0·73176	+0·73176	+0·73176
2	+1·22990	+1·21121	+1·15574	+1·06512	+0·86968	+0·61495	+0·42065	+0·21357	0·00000
3	+1·23766	+1·19102	+1·05670	+0·85088	+0·46412	+0·07735	−0·12844	−0·26277	−0·30941
4	−0·07073	−0·06598	−0·05286	−0·03445	−0·00625	+0·01105	+0·01319	+0·00856	0·00000
5	−2·41068	−2·16202	−1·51029	−0·69684	+0·22600	+0·35783	+0·10557	−0·17988	−0·30134
6	−2·33728	−1·99807	−1·16415	−0·26883	+0·36153	+0·06391	−0·19408	−0·20929	0·00000
7	+2·46520	+1·99072	+0·90966	−0·04634	−0·29371	+0·17394	+0·18124	−0·05268	−0·19259
8	+2·51362	+1·89911	+0·61803	−0·26095	−0·09373	+0·19362	−0·03380	−0·16560	0·00000
9	−4·71857	−3·30119	−0·63448	+0·65908	−0·18202	−0·07589	+0·27973	+0·02119	−0·25805
10	+3·44945	+2·20968	+0·13604	−0·46030	+0·25607	−0·14533	−0·08707	+0·16063	0·00000
11	−1·60550	−0·93004	+0·05779	+0·15650	−0·10103	+0·07855	−0·04713	−0·02813	+0·06585
12	+0·53438	+0·27604	−0·04819	−0·02496	+0·01156	−0·00581	+0·01987	−0·01627	0·00000
13	−0·13345	−0·06049	+0·01642	−0·00058	+0·00296	−0·00383	+0·00005	+0·00310	−0·00430
14	+0·02605	+0·01017	−0·00354	+0·00115	−0·00118	+0·00090	−0·00078	+0·00045	0·00000
15	−0·00413	−0·00136	+0·00054	−0·00027	+0·00017	−0·00003	+0·00008	−0·00010	+0·00011
16	+0·00055	+0·00015	−0·00006	+0·00004	−0·00001	−0·00001	+0·00001		
17	−0·00006	−0·00001	+0·00001						
A_1	−4·43777	−2·54161	+0·62811	+1·65419	+0·84825	+1·33968	+1·12286	+0·23249	−0·26797
A_2	+5·34594	+3·54231	+0·64101	+0·01682	+1·39767	+0·73328	+0·13799	−0·00795	0·00000

Table II.

$$(-1)^n \cdot \frac{2n+1}{n\,(n+1)} \cdot \frac{S_n^2}{|E_n|^2} \cdot P'_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1	− 0·91441	− 0·91441	−0·91441	−0·91441	−0·91441	−0·91441	−0·91441	−0·91441	−0·91441
2	+ 1·47323	+ 1·45086	+1·38440	+1·27585	+1·04174	+0·73661	+0·50387	+0·25583	0·00000
3	− 0·51278	− 0·49346	−0·43780	−0·35253	−0·19229	−0·03205	+0·05322	+0·10887	+0·12820
4	+ 4·49904	+ 4·19691	+3·36225	+2·19164	+0·39767	−0·70298	−0·83907	−0·54472	0·00000
5	− 4·42676	− 1·27959	−0·89386	−0·41242	+0·13376	+0·21178	+0·06248	−0·10646	−0·17835
6	+ 0·99175	+ 0·84781	+0·49397	+0·11407	−0·15340	−0·02712	+0·08235	+0·08881	0·00000
7	− 6·57582	− 5·31019	−2·42650	+0·12362	+0·78345	−0·46397	−0·48345	+0·14051	+0·51374
8	+ 7·67697	+ 5·80016	+1·88754	−0·79697	−0·28627	+0·59133	−0·10325	−0·50578	0·00000
9	− 4·20186	− 2·93968	−0·56500	+0·58691	−0·16208	−0·06758	+0·24910	+0·01887	−0·22979
10	+ 1·29223	+ 0·82779	+0·05096	−0·17244	+0·09593	−0·05444	−0·03262	+0·06018	0·00000
11	− 0·22869	− 0·13248	+0·00823	+0·02229	−0·01439	+0·01119	−0·00671	−0·00401	+0·00938
12	+ 0·02289	+ 0·01182	−0·00206	−0·00107	+0·00049	−0·00025	+0·00085	−0·00070	0·00000
13	− 0·00132	− 0·00060	+0·00016	−0·00001	+0·00003	−0·00004	0·00000	+0·00003	−0·00004
14	+ 0·00001	+ 0·00002	−0·00001						
15									
16									
17									
B_1	−13·86164	−11·07041	−5·22918	−0·94655	−0·36593	−1·25508	−1·03977	−0·75660	−0·67127
B_2	+15·95612	+13·13537	+7·17705	+2·61108	+1·09616	+0·54315	−0·38787	−0·64638	0·00000

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY
PHILOSOPHICAL TRANSACTIONS OF

TABLE III.

$$(-1)^n \frac{2n+1}{n(n+1)} \cdot \frac{S'_n C'_n}{|E'_n|^2} \cdot P'_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1	−0·73208	−0·73208	−0·73208	−0·73208	−0·73208	−0·73208	−0·73208	−0·73208	−0·73208
2	−1·22846	−1·20979	−1·15438	−1·06387	−0·86866	−0·61423	−0·42016	−0·21332	0·00000
3	−1·22031	−1·17433	−1·04189	−0·83896	−0·45761	−0·07627	+0·12664	+0·25908	+0·30508
4	+0·13013	+0·12139	+0·09725	+0·06339	+0·01150	−0·02033	−0·02427	−0·01576	0·00000
5	+2·47053	+2·21569	+1·54778	+0·71413	−0·23161	−0·36672	−0·10819	+0·18434	+0·30882
6	+2·14057	+1·82991	+1·06618	+0·24621	−0·33110	−0·05853	+0·17774	+0·19168	0·00000
7	−2·86418	−2·31292	−1·05689	+0·05384	+0·34124	−0·20209	−0·21057	+0·06120	+0·22376
8	−1·41534	−1·06933	−0·34799	+0·14693	+0·05278	−0·10902	+0·01903	+0·09325	0·00000
9	+4·16121	+2·91125	+0·55954	−0·58123	+0·16052	+0·06692	−0·24669	−0·01869	+0·22756
10	−4·59928	−2·94625	−0·18138	+0·61373	−0·34142	+0·19377	+0·11609	−0·21418	0·00000
11	+2·69718	+1·56243	−0·09709	−0·26291	+0·16973	−0·13196	+0·07918	+0·04725	−0·11063
12	−0·79823	−0·41233	+0·07199	+0·03728	−0·01726	+0·00868	−0·02968	+0·02430	0·00000
13	+0·17486	+0·07926	−0·02152	+0·00076	−0·00387	+0·00502	−0·00006	−0·00406	+0·00564
14	−0·03159	−0·01233	+0·00430	−0·00140	+0·00144	−0·00109	+0·00095	−0·00054	0·00000
15	+0·00479	+0·00157	−0·00063	+0·00032	−0·00019	+0·00004	−0·00009	+0·00012	−0·00013
16	−0·00062	−0·00017	+0·00007	−0·00004	+0·00001	+0·00001	−0·00001	+0·00000	0·00000
17	+0·00007	+0·00001	−0·00001						
A'_1	+4·69207	+2·55088	−0·84279	−1·64613	−0·75387	−1·43714	−1·09186	−0·20284	+0·22802
A'_2	−5·80282	−3·69890	−0·44396	+0·04223	−1·49271	−0·60074	−0·16031	−0·13457	0·00000

TABLE IV.

$$(-1)^n \frac{2n+1}{n(n+1)} \cdot \frac{S'^2_n P'_n}{|E'_n|^2}. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1	− 0·58704	− 0·58704	−0·58704	−0·58704	−0·58704	−0·58704	−0·58704	−0·58704	−0·58704
2	+ 1·01895	+ 1·00346	+0·95750	+0·88243	+0·72051	+0·50947	+0·34850	+0·17694	0·00000
3	− 3·00442	− 2·89121	−2·56513	−2·06552	−1·12665	−0·18778	+0·31180	+0·63787	+0·75109
4	+ 0·00376	+ 0·00351	+0·00281	+0·00183	+0·00033	−0·00059	−0·00070	−0·00046	0·00000
5	− 3·95795	− 3·54969	−2·47965	−1·14409	+0·37106	+0·58750	+0·17332	−0·29533	−0·49473
6	+ 5·69547	+ 4·86889	+2·83681	+0·65509	−0·88097	−0·15574	+0·47292	+0·51000	0·00000
7	− 1·32944	− 1·07357	−0·49057	+0·02499	+0·15839	−0·09380	−0·09774	+0·02841	+0·10386
8	+ 0·24259	+ 0·18329	+0·05965	−0·02518	−0·00905	+0·01869	−0·00326	−0·01598	0·00000
9	− 2·45947	− 1·72068	−0·33071	+0·34353	−0·09487	−0·03955	+0·14580	+0·01104	−0·13450
10	+ 2·71838	+ 1·74137	+0·10721	−0·36274	+0·20179	−0·11452	−0·06862	+0·12659	0·00000
11	− 0·67180	− 0·38916	+0·02418	+0·06548	−0·04227	+0·03287	−0·01972	−0·01177	+0·02755
12	+ 0·05118	+ 0·02644	−0·00462	−0·00239	+0·00111	−0·00056	+0·00190	−0·00156	0·00000
13	− 0·00227	− 0·00103	+0·00028	−0·00001	+0·00005	−0·00007	0·00000	+0·00005	−0·00007
14	+ 0·00007	+ 0·00003	−0·00001						
15									
16									
B'_1	−12·01239	−10·21238	−6·42864	−3·36266	−1·32133	−0·28787	−0·07358	−0·21677	−0·33384
B'_2	+ 9·73040	+ 7·82699	+3·96398	+1·14904	+0·03372	+0·25675	+0·75074	+0·79553	0·00000

Table V.

$$(-1)^n \cdot (2n+1) \cdot \frac{S_n C_n}{|E_n|^2} \cdot P_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1		+1·44129	+1·37528	+1·26745	+1·03488	+0·73176	+0·50055	+0·25414	0·00000
2		+2·34855	+2·02820	+1·53738	+0·61495	−0·30748	−0·79829	−1·11864	−1·22990
3		+2·25393	+1·64581	+0·80388	−0·43759	−1·08295	−1·02233	−0·61235	0·00000
4		−0·12069	−0·06719	−0·00332	+0·05747	+0·04089	+0·00054	−0·03761	−0·05305
5		−3·77990	−1·30897	+1·07647	+1·81113	−0·43317	−1·58172	−1·35488	0·00000
6		−3·29306	−0·33612	+1·74840	+0·69388	−1·51102	−0·97641	+0·61760	+1·46080
7		+3·03927	−0·52867	−2·02232	+0·62644	+1·10020	−0·73228	−1·39765	0·00000
8		+2·62343	−1·26605	−1·70310	+1·49982	−0·37020	−1·39766	+0·11718	+1·37465
9		−3·99003	+3·31902	+1·78904	−2·69464	+2·52819	+0·44907	−2·45014	0·00000
10		+2·21758	−2·76834	−0·04856	+0·79415	−1·29858	+1·51386	+0·44623	−1·69780
11		−0·70633	+1·28484	−0·51602	+0·33453	−0·20509	−0·59867	+0·68905	0·00000
12		+0·12885	−0·37711	+0·29199	−0·26368	+0·24983	−0·08421	−0·13970	+0·24110
13		−0·00675	+0·07160	−0·08184	+0·06388	−0·04425	+0·05977	−0·04122	0·00000
14		−0·00333	−0·00826	+0·01347	−0·00507	−0·00298	−0·00388	+0·00902	−0·01091
15		+0·00120	+0·00031	−0·00121	−0·00075	+0·00173	−0·00132	+0·00071	0·00000
16		−0·00024	+0·00009	+0·00000	+0·00024	−0·00016	+0·00019	−0·00021	+0·00022
17		+0·00003	−0·00002	+0·00002	−0·00003	−0·00001			
C_1		−1·74729	+5·85920	+2·31547	+0·73785	+2·59641	−2·92693	−4·91234	0·00000
C_2		+3·90109	−2·79478	+1·83626	+3·39176	−3·19970	−1·74586	−0·10613	+0·08511

Table VI.

$$(-1)^n (2n+1) \cdot \frac{S_n^{\,2}}{|E_n|^2} \cdot P_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1		− 1·80103	−1·71854	−1·58380	−1·29318	−0·91441	−0·62549	−0·31757	0·00000
2		+ 2·81320	+2·42946	+1·84154	+0·73662	−0·36831	−0·95622	−1·33995	−1·47323
3		− 0·93383	−0·68188	−0·33306	+0·18130	+0·44868	+0·42356	+0·25371	0·00000
4		+ 7·67733	+4·27386	+0·21089	−3·65545	−2·60100	−0·03419	+2·39260	+3·37430
5		− 2·23712	−0·77471	+0·63710	+1·07192	−0·25637	−0·93614	−0·80188	0·00000
6		+ 1·39730	+0·14262	−0·74187	−0·29442	+0·64115	+0·41430	−0·26206	−0·61984
7		− 8·10718	+1·41020	+5·39448	−1·67101	−2·93474	+1·95346	+3·72821	0·00000
8		+ 8·01235	−3·86670	−5·20152	+4·58070	−1·13066	−4·26866	+0·35787	+4·19840
9		− 3·55304	+2·95556	+1·59313	−2·39955	+2·25133	+0·39989	−2·18183	0·00000
10		+ 0·83075	−1·03705	−0·01819	+0·29750	−0·48648	+0·56675	+0·16717	−0·63602
11		− 0·10061	+0·18302	−0·07350	+0·04765	−0·02921	−0·08527	+0·09815	0·00000
12		+ 0·00552	−0·01615	+0·01251	−0·01129	+0·01070	−0·00361	−0·00598	+0·01033
13		− 0·00007	+0·00071	−0·00081	+0·00063	−0·00044	+0·00059	−0·00041	0·00000
14		− 0·00001	−0·00001	+0·00002	−0·00001	−0·00001	−0·00001	+0·00002	−0·00002
15									
16									
D_1		−16·73288	+1·37436	+5·63354	−4·06224	−1·43516	+1·13060	+0·77838	0·00000
D_2		+20·73644	+1·92603	−3·89662	+1·65365	−3·93461	−4·28164	+1·30967	+4·85392

Note.—The gaps in Tables V. to VIII. corresponding to $\theta = 0$ are left owing to the fact that the entries would be just double the corresponding entries in Tables I. to IV. respectively, as is shown at the end of § 2.

Table VII.

$$(-1)^n \cdot (2n+1) \cdot \frac{S'_n C'_n}{|E'_n|^2} \cdot P_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1		−1·44192	−1·37588	−1·26800	−1·03533	−0·73208	−0·50077	−0·25425	0·00000
2		−2·34579	−2·02582	−1·53557	−0·61423	+0·30711	+0·79735	+1·11733	+1·22848
3		−2·22234	−1·62274	−0·79261	+0·43145	+1·06778	+1·00800	+0·60377	0·00000
4		+0·22206	+0·12362	+0·00610	−0·10573	−0·07523	−0·00099	+0·06921	+0·09760
5		+3·87374	+1·34147	−1·10319	−1·85609	+0·44393	+1·62098	+1·38851	0·00000
6		+3·01592	+0·30783	−1·60125	−0·63548	+1·38385	+0·89423	−0·56563	−1·33786
7		−3·53118	+0·61423	+2·34963	−0·72783	−1·27826	+0·85080	+1·62387	0·00000
8		−1·47717	+0·71287	+0·95896	−0·84450	+0·20845	+0·78697	−0·06598	−0·77403
9		+3·51868	−2·92698	−1·57772	+2·37635	−2·22956	−0·39602	+2·16073	0·00000
10		−2·95679	+3·69114	+0·06474	−1·05886	+1·73145	−2·01716	−0·59498	+2·26371
11		+1·18662	−2·15849	+0·86690	−0·56200	+0·34455	+1·00575	−1·15752	0·00000
12		−0·19247	+0·56331	−0·43616	+0·39388	−0·37318	+0·12579	+0·20867	−0·36014
13		+0·00884	−0·09382	+0·10725	−0·08670	+0·05799	−0·07831	+0·05401	0·00000
14		+0·00404	+0·01002	−0·01633	+0·00615	+0·00361	+0·00471	−0·01094	+0·01323
15		−0·00139	−0·00036	+0·00141	+0·00087	−0·00201	+0·00153	−0·00082	0·00000
16		+0·00027	−0·00010	−0·00000	−0·00027	+0·00019	−0·00022	+0·00024	−0·00024
17		−0·00004	+0·00002	−0·00002	+0·00003	+0·00001	−0·00000	−0·00000	0·00000
C'_1		+1·39101	−6·22255	−1·41635	−1·45925	−2·32765	+3·51196	+4·41830	0·00000
C'_2		−3·72993	+3·38287	−2·55951	−2·85904	+3·18625	+0·59068	+0·15792	+1·13075

Table VIII.

$$(-1)^n \cdot (2n+1) \cdot \frac{(S'_n)^2}{|E'_n|^2} \cdot P_n. \quad \kappa a = 10.$$

n.	0°.	10°.	20°.	30°.	45°.	60°.	70°.	80°.	90°.
1		− 1·15625	−1·10329	−1·01679	−0·83021	−0·58704	−0·40156	−0·20388	0·00000
2		+ 1·94572	+1·68031	+1·27368	+0·50947	−0·25474	−0·66136	−0·92677	−1·01895
3		− 5·47142	−3·99521	−1·95142	+1·06223	+2·62888	+2·48170	+1·48648	0·00000
4		+ 0·00643	+0·00358	+0·00018	−0·00306	−0·00218	−0·00003	+0·00200	+0·00282
5		− 6·20597	−2·14912	+1·76738	+2·97358	−0·71120	−2·59693	−2·22449	0·00000
6		+ 8·02454	+0·81905	−4·26049	−1·69083	+3·68205	+2·37930	−1·50498	−3·55967
7		− 1·63904	+0·28510	+1·09061	−0·33783	−0·59332	+0·39491	+0·75374	0·00000
8		+ 0·25319	−0·12219	−0·16437	+0·14475	−0·03573	−0·13489	+0·01131	+0·13267
9		− 2·07970	+1·72998	+0·93250	−1·40453	+1·31777	+0·23407	−1·27709	0·00000
10		+ 1·74759	−2·18162	−0·03826	+0·62584	−1·02336	+1·19223	+0·35166	−1·33795
11		− 0·29624	+0·53763	−0·21592	+0·13998	−0·08582	−0·25051	+0·28831	0·00000
12		+ 0·01234	−0·03612	+0·02797	−0·02526	+0·02393	−0·00807	−0·01338	+0·02309
13		− 0·00011	+0·00122	−0·00139	+0·00108	−0·00075	+0·00101	−0·00070	0·00000
14		− 0·00001	−0·00002	+0·00004	−0·00001	+0·00001	−0·00001	+0·00002	−0·00003
15									
16									
D'_1		−16·84873	−4·69369	+0·60497	+1·60430	+1·96852	−0·13731	−1·17763	0·00000
D'_2		+11·98980	+0·16299	−3·16125	−0·43910	+2·38996	+2·76717	−2·08014	−5·75802

Note.—The gaps in Tables V. to VIII. corresponding to $\theta = 0$ are left owing to the fact that the entries would be just double the corresponding entries in Tables I. to IV. respectively, as is shown at the end of § 2.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE X.

θ.	κa = 1.				κa = 2.			
	Y_1.	Y_2.	Z_1.	Z_2.	Y_1.	Y_2.	Z_1.	Z_2.
0°	−0·87964	+0·36828	+0·87964	−0·36828	−0·01686	−1·00392	+0·01686	+1·00392
10	−0·87409	+0·36177	+0·87833	−0·36931	−0·05133	−0·97319	+0·03713	+0·98979
20	−0·85739	+0·34240	+0·87421	−0·37235	−0·15388	−0·88511	+0·09780	+0·94713
30	−0·82898	+0·31068	+0·86695	−0·37735	−0·32775	−0·75189	+0·19180	+0·87470
45	−0·76305	+0·24213	+0·84886	−0·38814	−0·67558	−0·50959	+0·41394	+0·70713
60	−0·66730	+0·15213	+0·82006	−0·40234	−1·10637	−0·28864	+0·68944	+0·46459
70	−0·58650	+0·08407	+0·79396	−0·41326	−1·39254	−0·19417	+0·88593	+0·26063
80	−0·49274	+0·01031	+0·76198	−0·42497	−1·63685	−0·16609	+1·07223	+0·02454
90	−0·38750	−0·06572	+0·72430	−0·43714	−1·80040	−0·21751	+1·22758	−0·23984
100	−0·27326	−0·14203	+0·68166	−0·44941	−1·84751	−0·35271	+1·33131	−0·52610
110	−0·15360	−0·21597	+0·63534	−0·46136	−1·75394	−0·56577	+1·36711	−0·82519
120	−0·03310	−0·28649	+0·58728	−0·47266	−1·51443	−0·84086	+1·32726	−1·12573
135	+0·13733	−0·37819	+0·51714	−0·48762	−0·93146	−1·31449	+1·14028	−1·54979
150	+0·27800	−0·44904	+0·45767	−0·49917	−0·23151	−1·76695	+0·86400	−1·89896
160	+0·34623	−0·48198	+0·42835	−0·50453	+0·16600	−2·00347	+0·70070	−2·06803
170	+0·38897	−0·50215	+0·40985	−0·50783	+0·44169	−2·15677	+0·58211	−2·17383
180	+0·40352	−0·50896	+0·40352	−0·50896	+0·53950	−2·20986	+0·53950	−2·20986

θ.	κa = 9.				κa = 10.			
	Y_1.	Y_2.	Z_1.	Z_2.	Y_1.	Y_2.	Z_1.	Z_2.
0°	+2·92118	− 2·92969	−2·92118	+ 2·92969	+2·0189	+ 4·3765	−2·0189	− 4·3765
10	+2·85178	− 3·44035	−2·74647	+ 3·25040	+2·2090	+ 4·5747	−2·3163	− 4·3554
20	+2·39601	− 4·21198	−2·11542	+ 3·96256	+2·8996	+ 4·1626	−3·1586	− 3·9348
30	+0·93972	− 4·34162	−0·77288	+ 4·50196	+4·2579	+ 2·3038	−4·3085	− 2·5090
45	−2·79263	− 3·71421	+2·55340	+ 3·69379	+4·9757	− 1·3455	−4·7881	+ 1·6373
60	−4·11873	− 0·09190	+4·54564	− 0·28764	+0·1954	− 5·0860	−0·3981	+ 4·9827
70	−2·65205	+ 3·87786	+2·71046	− 3·61300	−3·2701	− 3·8259	+3·8519	+ 3·3399
80	+0·44242	+ 4·55372	−1·41433	+ 4·42270	−4·4103	+ 1·9553	+4·7201	− 1·7529
90	+4·07803	+ 0·45069	−4·49231	− 0·87232	−1·3987	+ 5·0868	+0·1429	− 5·1878
100	+3·98050	− 3·34098	−2·63233	+ 3·98232	+4·5127	+ 0·9681	−4·9162	− 1·5245
110	−2·08115	− 2·44198	+2·93266	+ 3·77514	+4·2248	− 3·2744	−2·4495	+ 4·8109
120	−5·97919	+ 0·37778	+4·53082	− 2·25240	−4·4893	− 1·9474	+4·6565	+ 2·8540
135	+4·89401	+ 2·22161	−4·61842	− 2·86833	+0·3279	+ 1·5395	−1·5266	− 6·0371
150	+0·17682	+ 1·81954	+0·68260	+ 7·60056	+4·2427	+ 4·1158	−2·6272	+ 8·0995
160	−9·31245	−10·01990	+8·06560	− 3·40086	−9·2435	− 7·4971	+8·2673	+ 0·7136
170	−5·30982	−30·84685	+5·13925	−28·36546	−7·1178	−35·2790	+6·5929	−31·6707
180	+0·74010	−41·89187	+0·74010	−41·89187	+0·7111	−51·5606	+0·7111	−51·5606

TABLE XI.

θ.	$\kappa a = 1$.		$\kappa a = 2$.		$\kappa a = 9$.		$\kappa a = 10$.	
	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
0°	0·9094	0·9094	1·0081	1·0081	17·12	17·12	23·23	23·23
10	0·8949	0·9079	0·9497	0·9811	19·97	18·11	25·81	24·33
20	0·8524	0·9029	0·8071	0·9066	23·48	20·18	25·74	25·46
30	0·7837	0·8940	0·6728	0·8019	19·73	20·86	23·44	24·86
45	0·6409	0·8712	0·7161	0·6714	21·59	20·16	26·57	25·61
60	0·4684	0·8344	1·3074	0·6912	16·97	20·75	25·91	24·99
70	0·3511	0·8012	1·9769	0·8528	22·07	20·40	25·33	25·99
80	0·2429	0·7612	2·7069	1·1503	20·93	21·56	23·27	25·35
90	0·1545	0·7157	3·2887	1·5645	16·83	20·94	27·83	26·93
100	0·0948	0·6666	3·5377	2·0492	27·01	22·79	21·30	26·49
110	0·0702	0·6165	3·3964	2·5499	10·29	22·85	28·57	29·14
120	0·0832	0·5683	3·0005	3·0289	35·89	25·60	23·95	29·83
135	0·1619	0·5052	2·5955	3·7021	28·89	29·56	2·48	38·78
150	0·2789	0·4586	3·1757	4·3525	3·34	58·23	34·94	72·50
160	0·3522	0·4380	4·0415	4·7677	187·1	76·62	141·65	68·87
170	0·4035	0·4259	4·8467	5·0644	979·7	831·0	1295	1046
180	0·4219	0·4219	5·1745	5·1745	1755	1755	2659	2659

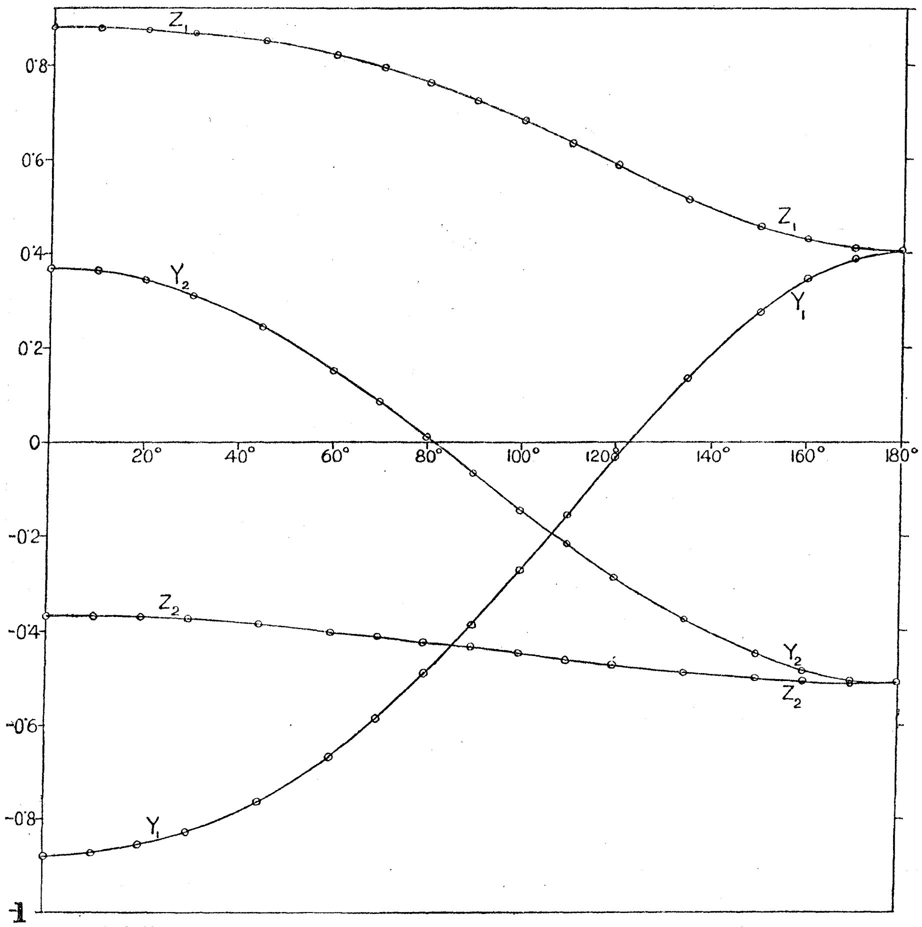

Fig. 1. Table X. ($\kappa a = 1$).

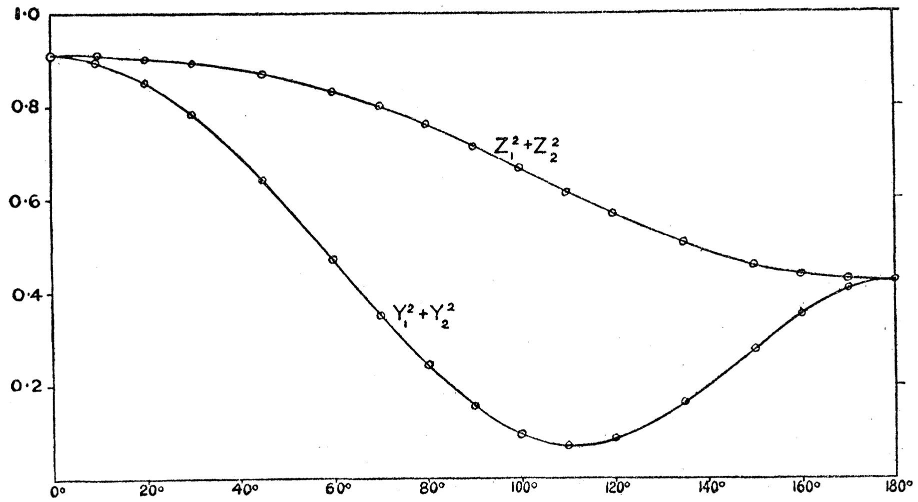

Fig. 2. Table XI. ($\kappa a = 1$).

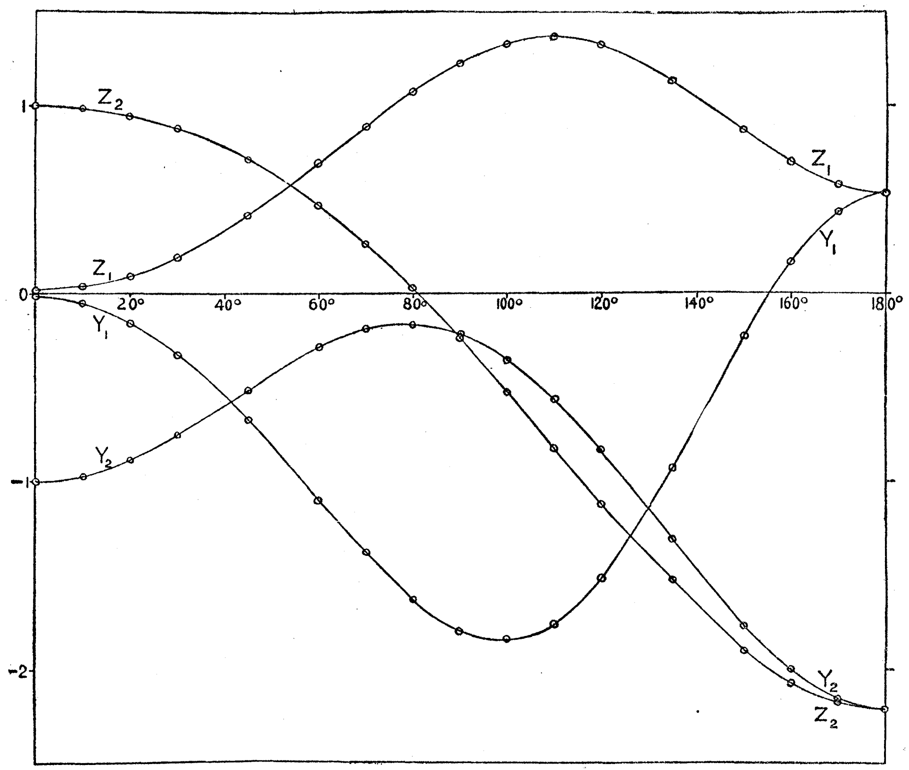

Fig. 3. Table X. ($\kappa a = 2$).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

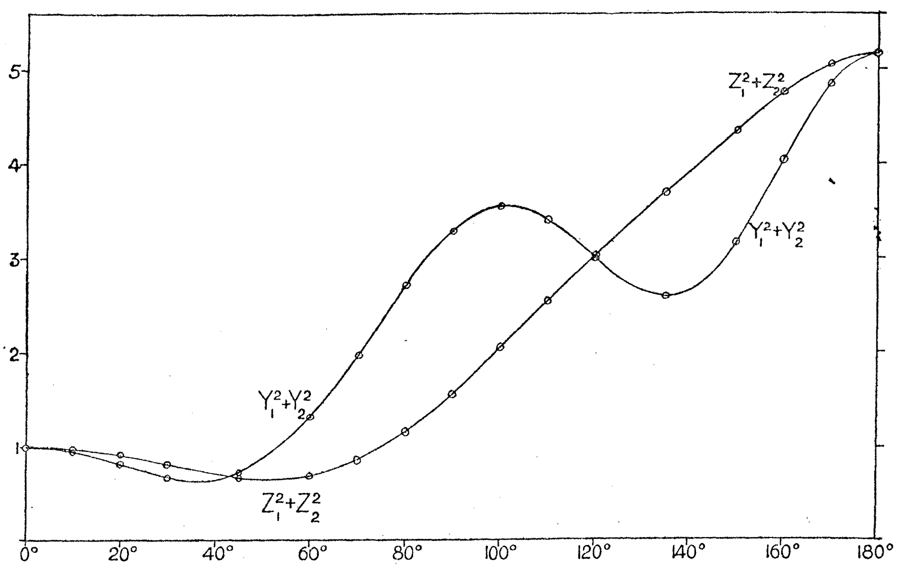

Fig. 4. Table XI. ($\kappa a = 2$).

Analysis of Results and Interpolation.

(*By* A. T. Doodson.)

4. A consideration of the results tabulated up to this point shows that for $\kappa a = 9$, 10, they are not adequate for the complete representation of $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$ throughout the range of θ. Owing to the excessive labour involved in further computations from the original series, other methods were tried. The following sections are devoted to the discussion and presentation of this work.

For values of θ, except those "near" 180°, the first approximation for large values of κa, gives

$$\left.\begin{aligned} Y_1 = -Z_1 &= \tfrac{1}{2}\kappa a \cos \Theta, \\ Y_2 = -Z_2 &= \tfrac{1}{2}\kappa a \sin \Theta, \\ \Theta &= 2\kappa a \cos \tfrac{1}{2}\,\theta. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad *(10)$$

where

Consider then the functions

$$\left.\begin{aligned} \eta_1 &= Y_1 - \tfrac{1}{2}\kappa a \cos \Theta, & \zeta_1 &= Z_1 + \tfrac{1}{2}\kappa a \cos \Theta, \\ \eta_2 &= Y_2 - \tfrac{1}{2}\kappa a \sin \Theta, & \zeta_2 &= Z_2 + \tfrac{1}{2}\kappa a \sin \Theta, \end{aligned}\right\} \quad \ldots\ldots \quad (11)$$

which express the amounts by which the functions Y_1, Y_2, Z_1, Z_2 differ from the values given by (10). These functions and their derivatives are calculable from the

* Bromwich, *loc. cit.*

preceding tables, at the values of θ taken. Tables XX. to XXIII.* give steps in the necessary computations, and figs. 5 to 12 show the results plotted against θ from 0° to 150°. These curves show that for this range the first approximation is very good for Z_1, Z_2. In fact, the curves for ζ_1, ζ_2 were easily drawn and gave interpolated values of Z_1, Z_2 to three places of decimals, the last figure being approximate. The functions η_1, η_2 have a greater number of oscillations and greater amplitudes, and there was in places some little doubt as to the exact forms of the curves. By careful examination and comparison of corresponding curves for the two cases of $\kappa a = 9$ and 10, this difficulty was overcome. Such a comparison was more fruitful when the curves were plotted with Θ as abscissæ, and this was done with η_1, η_2, ζ_1, ζ_2, though the curves are not given below. Corresponding curves are very similar but are not in phase with each other. In the range $\theta = 0°$ to 90°, it was found possible to obtain Y_1, Y_2 to at least two places of decimals, with very little doubt.

It was the work just described which led to the detection of errors mentioned in §§ 1, 3, How this was possible becomes evident on an examination of the formulæ and numbers. For example, an error in Y_1 or Z_1 generally leads to a larger relative error in $\partial Y_1/\partial\theta$ and $\partial Z_1/\partial\theta$, to a much larger relative error in η_1 or ζ_1, and to a very much larger relative error in $\partial\eta_1/\partial\theta$ and $\partial\eta_1/\partial\theta$. Thus, an error might pass unsuspected in the graph of Y_1 (for example) but render the drawing of the graph of η_1 impossible. Irregularities could also be detected in the curves plotted with Θ as abscissæ. By these means it was possible easily to detect errors in Y_1, Y_2, Z_1, Z_2, of the magnitude 0·01. The actual finding of the errors involved much patient revision of the summing of the series. Their existence had previously been entirely unsuspected.

In this part of the work it was necessary to calculate a number of values of $\cos\Theta$ and $\sin\Theta$. The method adopted was to find θ such that $\Theta/\pi = r+\theta_1/180$, where r is an integer and $\theta_1 = 0°, 10°, 20° \ldots 170°$. This yielded about eighteen points on each undulation, and a selection from these was made according to circumstances.

* Tables XXII., XXIII., and figs. 5 to 12 are not printed.

Table XXII. gives η_1, η_2, &c. ($\kappa a = 9, 10$).
,, XXIII. ,, $\partial\eta_1/\partial\theta$, $\partial\eta_2/\partial\theta$, &c. ($\kappa a = 9\ \ 10$).
Fig. 5 gives the graph of η_1 ($\kappa a = 9$).
,, 6 ,, ,, ,, η_1 ($\kappa a = 10$).
,, 7 ,, ,, ,, η_2 ($\kappa a = 9$).
,, 8 ,, ,, ,, η_2 ($\kappa a = 10$).
,, 9 ,, ,, ,, ζ_1 ($\kappa a = 9$).
,, 10 ,, ,, ,, ζ_1 ($\kappa a = 10$)
,, 11 ,, ,, ,, ζ_2 ($\kappa a = 9$).
,, 12 ,, ,, ,, ζ_2 ($\kappa a = 10$).

TABLE XX.

$\kappa a = 9$.			θ.	$\kappa a = 10$.		
$180\Theta/\pi$.	sin Θ.	cos Θ.		$180\Theta/\pi$.	sin Θ.	cos Θ.
1031·3240	−0·75099	+0·66032	0°	1145·9156	+0·91295	+0·40808
1027·3995	−0·79442	+0·60737	10	1141·5550	+0·87927	+0·47631
1015·6558	−0·90141	+0·43296	20	1128·5066	+0·74903	+0·66253
996·1825	−0·99418	+0·10770	30	1106·8694	+0·45196	+0·89204
952·8189	−0·79673	−0·60434	45	1058·6878	−0·36345	+0·93161
893·1527	+0·11922	−0·99287	60	992·3918	−0·99913	+0·04173
844·8112	+0·82104	−0·57087	70	938·6791	−0·62496	−0·78066
790·0400	+0·93993	+0·34136	80	877·8222	+0·37748	−0·92602
729·2561	+0·16085	+0·98698	90	810·2846	+0·99999	−0·00497
662·9223	−0·83941	+0·54350	100	736·5803	+0·28536	+0·95842
591·5431	−0·78308	−0·62192	110	657·2702	−0·88886	+0·45819
515·6620	+0·41212	−0·91113	120	572·9577	−0·54402	−0·83907
394·6706	+0·56880	+0·82244	135	438·5229	+0·98000	+0·19898
266·9263	−0·99856	−0·05362	150	296·5847	−0·89427	+0·44752
179·0876	+0·01592	−0·99987	160	198·9861	−0·32534	−0·94560
89·8858	+1·00000	+0·00199	170	99·8731	+0·98519	−0·17125
0·0000	0·00000	+1·00000	180	0·0000	0·00000	+1·00000

TABLE XXI.

$\kappa a = 9$.		θ.	$\kappa a = 10$.	
$\frac{\partial}{\partial\theta}$. sin Θ.	$\frac{\partial}{\partial\theta}$. cos Θ.		$\frac{\partial}{\partial\theta}$. sin Θ.	$\frac{\partial}{\partial\theta}$. cos Θ.
0·00000	0·00000	0°	0·00000	0·00000
−0·00832	−0·01088	10	−0·00725	+0·01337
−0·01181	−0·02459	20	−0·02008	+0·02270
−0·00438	−0·04042	30	−0·04029	+0·02042
+0·03633	−0·04789	45	−0·06222	−0·02428
+0·07798	+0·00936	60	−0·00364	−0·08719
+0·05143	+0·07397	70	+0·07815	−0·06256
−0·03447	+0·09491	80	+0·10389	+0·04235
−0·10963	+0·01787	90	+0·00061	+0·12341
−0·06540	−0·10100	100	−0·12814	+0·03815
+0·08002	−0·10076	110	−0·06551	−0·12708
+0·12395	+0·05606	120	+0·12683	−0·08223
−0·11936	+0·08255	135	−0·03208	+0·15802
+0·00814	−0·15151	150	−0·07544	−0·15076
+0·15467	+0·00246	160	+0·16253	−0·05592
−0·00031	+0·15648	170	+0·02977	+0·17129
−0·15708	+0·00000	180	+0·17453	+0·00000

5. There remains the discussion of Y_1, Y_2 for the range $\theta = 90°$ to $180°$, and of Z_1, Z_2 for the range $\theta = 120°$ to $180°$. For portions of these ranges a method was used which consists in decomposing η_1, η_2, ζ_1, ζ_2 into functions for which graphical interpolation is easier.

For both $\kappa a = 9$ and 10, take

$$\left.\begin{aligned} \eta_1 &= \alpha_1 \cos\Theta + \beta_1 \sin\Theta, \qquad & \zeta_1 &= \gamma_1 \cos\Theta + \delta_1 \sin\Theta, \\ \eta_2 &= \alpha_2 \cos\Theta + \beta_2 \sin\Theta, \qquad & \zeta_2 &= \gamma_2 \cos\Theta + \delta_2 \sin\Theta, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2 are functions of θ only.

These functions, thus defined, are perfectly determinate, but they can only be used for $\kappa a = 9$, 10. Together with their derivatives, they are calculable for the tabulated values of θ. We have, in fact,

$$\left.\begin{aligned} \alpha_1 &= \frac{\Delta(\eta_1/\sin\Theta)}{\Delta(\cot\Theta)}, \qquad & \beta_1 &= \frac{\Delta(\eta_1/\cos\Theta)}{\Delta(\tan\Theta)}, \\ \alpha_2 &= \frac{\Delta(\eta_2/\cos\Theta)}{\Delta(\tan\Theta)}, \qquad & \beta_2 &= \frac{\Delta(\eta_2/\sin\Theta)}{\Delta(\cot\Theta)}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

with similar formulæ for γ_1, γ_2, δ_1, δ_2, where $\Delta F(\kappa a)$ denotes $F(10) - F(9)$. Also,

$$\left.\begin{aligned} \frac{\partial\alpha_1}{\partial\theta} &= \left[\Delta\left(\frac{\partial\eta_1}{\partial\theta}\Big/\sin\Theta\right) + \frac{\pi}{180}\sin\tfrac{1}{2}\theta\,\{\alpha_1 + \beta_1\Delta(\kappa a\cot\Theta)\}\right]\Big/\Delta(\cot\Theta), \\ \frac{\partial\beta_1}{\partial\theta} &= \left[\Delta\left(\frac{\partial\eta_1}{\partial\theta}\Big/\cos\Theta\right) - \frac{\pi}{180}\sin\tfrac{1}{2}\theta\,\{\beta_1 + \alpha_1\Delta(\kappa a\tan\Theta)\}\right]\Big/\Delta(\tan\Theta), \end{aligned}\right\} \quad . \quad (14)$$

with similar formulæ for the remaining cases.

The values of the functions themselves are given in Table XXIV. for the range $\theta = 0°$ to $150°$, and are plotted by means of small circles in figs. 13 to 16. As will be seen, it is possible to draw smooth curves with few undulations approximately through these points. The complete graphs of α_1, α_2, &c., differ from the curves of figs. 13 to 16 by a number of ripples. An attempt has only been made to draw these ripples in those parts of the curves actually required for interpolation. For this the gradients were necessary, and the values of the derivatives used are given in Table XXV. With the limited data of Tables XXIV., XXV. the drawing of the ripples is not perfectly determinate, but it was assumed that the graphs of η_1, η_2, &c. (and of Y_1, Y_2, &c.), are without ripples, and after a number of trials the ripples of the graphs of α_1, α_2, &c., were drawn in such a way as to secure this, but they are not given in the figures.

TABLE XXIV.

θ.	α_1.	β_1.	α_2.	β_2.	γ_1.	δ_1.	γ_2.	δ_2.
0°	−0·068	+0·007	−0·340	+0·296	+0·068	−0·007	+0·340	−0·296
10	−0·036	−0·183	+0·049	+0·285	+0·043	+0·051	+0·197	−0·277
20	−0·040	−0·516	+0·308	+0·282	+0·015	+0·192	−0·022	−0·261
30	+0·005	−0·459	−0·121	+0·115	+0·022	+0·293	−0·056	−0·250
45	+0·290	−0·129	−0·172	+0·440	−0·035	+0·242	+0·008	−0·190
60	−0·352	−0·001	+0·116	+0·648	−0·057	+0·187	+0·003	−0·250
70	−0·469	−0·428	+0·545	+0·461	−0·046	+0·140	−0·059	−0·228
80	−0·620	−0·938	+0·324	+0·059	−0·039	+0·143	−0·131	−0·199
90	−0·144	−1·38	+0·085	−0·291	−0·072	+0·118	−0·187	−0·120
100	+0·212	−1·69	−0·696	−0·272	−0·164	+0·116	−0·260	−0·025
110	+0·960	−1·68	−1·34	−0·050	−0·267	+0·041	−0·377	+0·070
120	+1·360	−1·56	−2·31	+0·576	−0·505	−0·070	−0·541	+0·261
135	+2·23	−1·14	−3·90	+2·28	−0·861	−0·368	−1·26	+0·495
150	+3·30	−0·59	−6·81	+5·83	−1·586	−0·356	−3·20	+1·71

TABLE XXV.

θ.	$\frac{\partial\alpha_1}{\partial\theta}$.	$\frac{\partial\beta_1}{\partial\theta}$.	$\frac{\partial\alpha_2}{\partial\theta}$.	$\frac{\partial\beta_2}{\partial\theta}$.	$\frac{\partial\gamma_1}{\partial\theta}$.	$\frac{\partial\delta_1}{\partial\theta}$.	$\frac{\partial\gamma_2}{\partial\theta}$.	$\frac{\partial\delta_2}{\partial\theta}$.
90°	+0·052	−0·062	−0·043	+0·0032				
100	+0·049	−0·009	−0·082	−0·0099				
110	+0·066	−0·020	−0·075	+0·065		−0·014		
120	+0·050	+0·055	−0·087	+0·056	−0·026	−0·005		+0·019
135	+0·022	+0·049	−0·148	+0·089	−0·055	−0·036	−0·064	+0·023
150	+0·034	+0·010	−0·400	+0·148	−0·027	−0·049	−0·277	+0·124

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

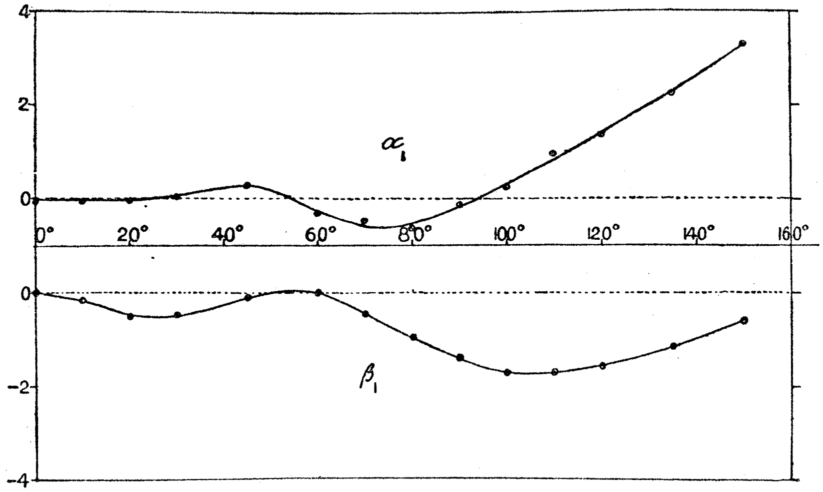

Fig. 13. Table XXIV.

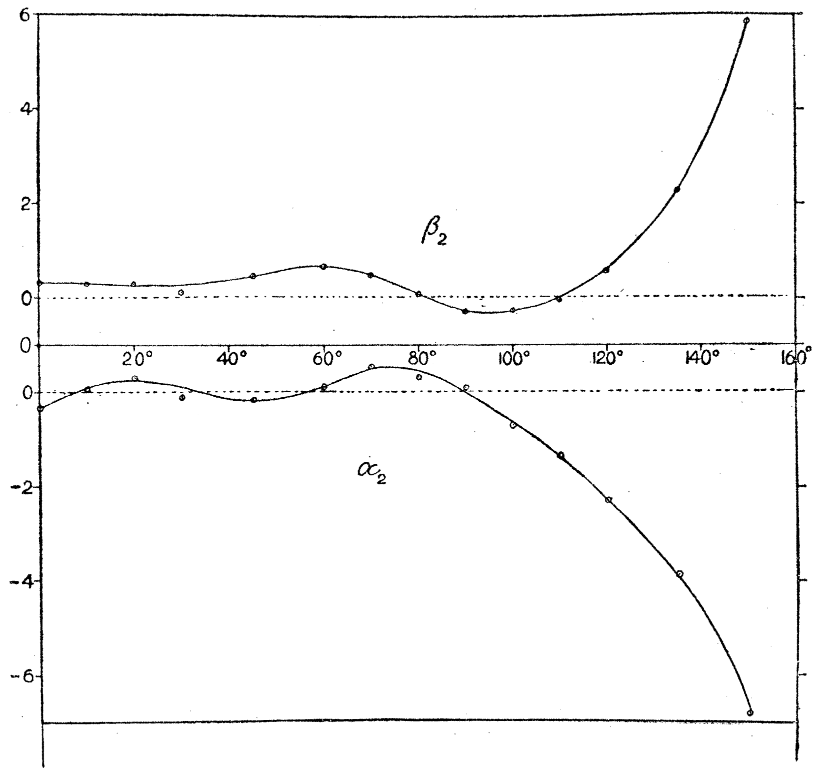

Fig. 14. Table XXIV.

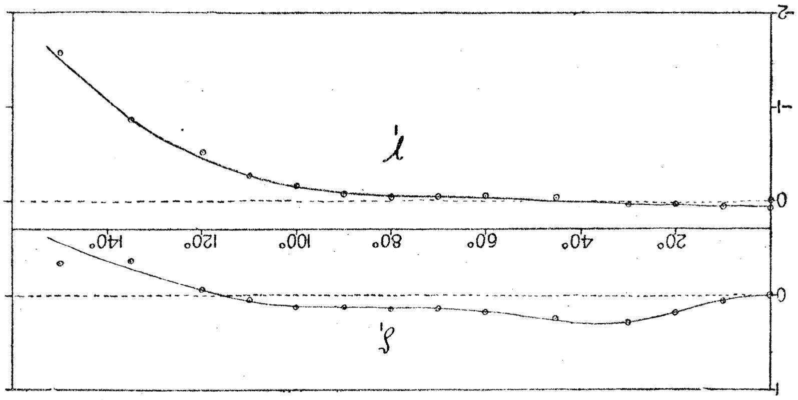

Fig. 15. Table XXIV.

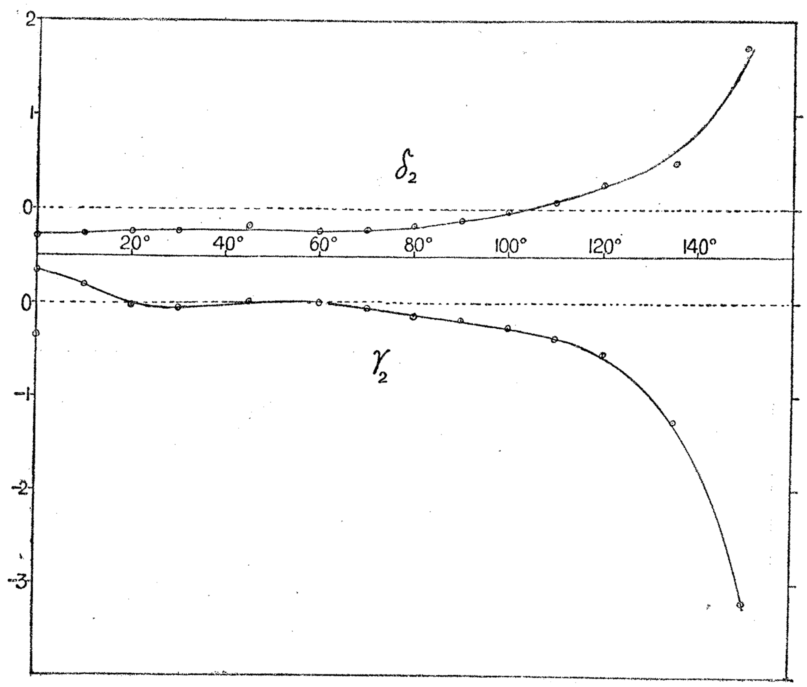

Fig. 16. Table XXIV.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In using the functions α_1, α_2, &c., considerations of the possible forms for the theoretical higher approximations for large values of κa have been continually in mind. Take Y_1, for example; the first approximation being $Y_1 = \frac{1}{2}\kappa a \cos \Theta$, let us assume an expression of the form

$$Y_1 = A(\theta, \kappa a) \cos \{\Phi(\theta, \kappa a)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Then in the first approximation, we have

$$A(\theta, \kappa a) = \tfrac{1}{2}\kappa a, \qquad \Phi(\theta, \kappa a) = \Theta.$$

Let us now suppose that higher approximations are got by taking

$$A(\theta, \kappa a) = \tfrac{1}{2}\kappa a + A_0(\theta) + \frac{A_1(\theta)}{\kappa a}, \qquad \Phi(\theta, \kappa a) = \Theta + \frac{\Phi_1(\theta)}{\kappa a} + \frac{\Phi_2(\theta)}{(\kappa a)^2}, \quad . \quad (16)$$

where the new functions of θ are small compared with κa. The omission of a term independent of κa in $\Phi(\theta, \kappa a)$ is necessary if $Y_1/\frac{1}{2}\kappa a \cos \Theta \to 1$ as $\kappa a \to \infty$.

To the same order of approximation, (16) yields

$$\eta_1 = \left(a_0 + \frac{a_1}{\kappa a}\right) \cos \Theta + \left(b_0 + \frac{b_1}{\kappa a}\right) \sin \Theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where a_0, a_1, b_0, b_1 are functions of θ, being calculable in terms of those in (16).

We then obtain, from (13),

$$\alpha_1 = a_0 + \frac{a_1}{180}\left\{19 + \frac{\sin(38 \cos \frac{1}{2}\theta)}{\sin(2 \cos \frac{1}{2}\theta)}\right\} + \frac{b_1}{90}\frac{\sin(18 \cos \frac{1}{2}\theta) \sin(20 \cos \frac{1}{2}\theta)}{\sin(2 \cos \frac{1}{2}\theta)}, \quad . \quad (18)$$

and a similar expression for β_1.

It is interesting to compare this hypothetical approximate expression for α_1 with what is known about the exact value of this function. Since the coefficients of a_1 and b_1 in (18) are rapidly oscillating functions of θ with small amplitude, the suggestion immediately arises that the smooth curve α_1 of fig. 13 is the graph of a_0, or perhaps of $a_0 + \dfrac{19}{180}a_1$, and that the ripples of the same figure are approximately represented by the remainder of (18).

From $\theta = 150°$ to $\theta = 180°$ simple graphical interpolation of Y_1, Y_2, Z_1, Z_2 has been relied upon.

Table XXVI. contains all the interpolated values of Y_1, Y_2, Z_1, Z_2, $Y_1^2 + Y_2^2$, $Z_1^2 + Z_2^2$ that have been used.

TABLE XXVI.

$\kappa a = 9$.						
θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
° 5·81	2·893	− 3·096	−2·863	+ 3·038	18·0	17·43
12·70	2·791	− 3·680	−2·631	+ 3·426	21·3	18·66
17·00	2·610	− 4·035	−2·369	+ 3·737	23·1	19·58
20·42	2·361	− 4·233	−2·077	+ 3·992	23·5	20·25
23·34	2·043	− 4·337	−1·760	+ 4·191	23·0	20·66
25·95	1·678	− 4·367	−1·420	+ 4·343	21·9	20·88
28·31	1·272	− 4·360	−1·059	+ 4·449	20·6	20·92
30·50	0·830	− 4·333	−0·683	+ 4·513	19·5	20·83
32·55	0·370	− 4·284	−0·293	+ 4·540	18·5	20·70
34·48	−0·102	− 4·241	+0·102	+ 4·532	18·0	20·55
36·30	−0·562	− 4·182	+0·496	+ 4·488	17·8	20·39
38·05	−1·015	− 4·114	+0·895	+ 4·412	18·0	20·27
39·71	−1·459	− 4·036	+1·286	+ 4·299	18·4	20·13
41·32	−1·878	− 3·960	+1·672	+ 4·158	19·2	20·09
42·86	−2·270	− 3·875	+2·045	+ 3·987	20·2	20·08
44·36	−2·623	− 3·783	+2·414	+ 3·788	21·2	20·18
45·81	−2·952	− 3·641	+2·739	+ 3·565	22·0	20·21
47·22	−3·234	− 3·493	+3·063	+ 3·318	22·7	20·39
48·58	−3·487	− 3·312	+3·352	+ 3·047	23·1	20·52
49·92	−3·696	− 3·105	+3·616	+ 2·751	23·3	20·64
51·22	−3·869	− 2·870	+3·854	+ 2·440	23·2	20·81
52·49	−3·998	− 2·580	+4·056	+ 2·106	22·6	20·89
53·73	−4·081	− 2·251	+4·225	+ 1·759	21·7	20·95
54·95	−4·169	− 1·893	+4·364	+ 1·396	21·0	20·99
56·14	−4·203	− 1·511	+4·465	+ 1·022	19·9	20·98
57·31	−4·207	− 1·110	+4·531	+ 0·639	18·9	20·94
58·47	−4·198	− 0·686	+4·560	+ 0·250	18·1	20·86
60·70	−4·073	+ 0·201	+4·515	− 0·535	16·6	20·67
62·86	−3·872	+ 1·099	+4·330	− 1·309	16·2	20·46
64·96	−3·581	+ 2·000	+4·016	− 2·048	16·8	20·32
67·00	−3·247	+ 2·826	+3·577	− 2·733	18·5	20·26
68·99	−2·873	+ 3·551	+3·032	− 3·337	20·9	20·34
70·93	−2·439	+ 4·137	+2·395	− 3·847	23·1	20·54
72·82	−1·967	+ 4·589	+1·688	− 4·234	24·9	20·78
74·68	−1·427	+ 4·845	+0·930	− 4·498	25·5	21·10
76·49	−0·829	+ 4·917	+0·143	− 4·615	24·9	21·32
78·26	−0·217	+ 4·820	−0·645	− 4·587	23·3	21·46
80·00	+0·442	+ 4·554	−1·418	− 4·421	21·6	21·56
81·72	+1·123	+ 4·123	−2·146	− 4·109	18·3	21·49

TABLE XXVI. (continued).

$\kappa a = 9.$

θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2 + Y_2^2$.	$Z_1^2 + Z_2^2$.
° 83·41	+1·784	+ 3·541	−2·812	− 3·671	15·7	21·38
85·06	+2·415	+ 2·873	−3·392	− 3·116	14·1	21·22
86·69	+3·004	+ 2·116	−3·872	− 2·462	13·5	21·05
88·30	+3·574	+ 1·316	−4·238	− 1·727	14·5	20·94
89·88	+4·034	+ 0·514	−4·477	− 0·934	16·5	20·92
91·45	+4·410	− 0·281	−4·585	− 0·108	19·5	21·03
92·99	4·68	− 1·03	−4·552	0·729	23·0	21·25
94·51	4·81	− 1·71	−4·380	1·540	26·1	21·56
96·01	4·81	− 2·31	−4·068	2·300	28·5	21·84
97·50	4·65	− 2·78	−3·632	3·006	29·4	22·23
98·97	4·31	− 3·16	−3·081	3·615	28·6	22·56
100·42	3·84	− 3·41	−2·431	4·112	26·4	22·82
101·86	3·21	− 3·57	−1·709	4·485	23·1	23·04
103·29	2·48	− 3·59	−0·923	4·716	19·0	23·09
104·70	1·62	− 3·51	−0·100	4·800	14·9	23·05
106·09	0·71	− 3·32	+0·735	4·734	11·5	22·95
107·48	−0·27	− 3·07	1·556	4·527	9·5	22·92
108·85	−1·25	− 2·75	2·324	4·177	9·1	22·85
110·21	−2·27	− 2·39	3·040	3·694	10·9	22·89
111·56	−3·17	− 1·99	3·661	3·078	14·0	22·88
112·90	−3·98	− 1·60	4·181	2·360	18·4	23·05
114·22	−4·72	− 1·19	4·580	1·559	23·7	23·41
115·54	−5·31	− 0·79	4·834	0·704	28·8	23·86
116·85	−5·75	− 0·41	4·937	− 0·170	33·2	24·40
118·15	−5·99	− 0·07	4·879	− 1·052	35·9	24·91
119·44	−6·04	+ 0·25	4·669	− 1·905	36·5	25·43
120·73	−5·88	0·52	4·31	− 2·76	34·8	26·2
122·00	−5·54	0·76	3·82	− 3·50	31·3	26·8
123·27	−5·01	0·93	3·17	− 4·16	26·0	27·4
124·53	−4·31	1·10	2·42	− 4·67	19·8	27·7
125·78	−3·46	1·24	1·59	− 5·06	13·5	28·1
127·02	−2·47	1·41	0·70	− 5·31	8·1	28·7
128·26	−1·37	1·55	−0·21	− 5·40	4·3	29·2
129·49	−0·20	1·67	−1·10	− 5·30	2·8	29·3
130·72	+1·02	1·83	−1·99	− 5·05	4·4	29·5
131·94	2·22	1·95	−2·84	− 4·62	8·7	29·4
133·15	3·36	2·05	−3·63	− 4·04	15·5	29·5
134·36	4·39	2·15	−4·30	− 3·31	23·9	29·5
135·56	5·31	2·28	−4·85	− 2·47	33·4	29·6

TABLE XXVI. (continued).

$\kappa a = 9$.

θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
136·76°	6·02	2·55	−5·27	−1·49	42·7	30·0
137·95	6·56	2·77	−5·52	−0·44	50·7	30·7
139·14	6·89	3·00	−5·60	+0·67	56·5	31·8
140·32	6·99	3·16	−5·52	1·81	58·9	33·8
141·50	6·90	3·35	−5·29	2·93	58·8	36·6
142·68	6·56	3·41	−4·85	4·04	54·7	39·8
143·85	6·04	3·40	−4·23	5·02	48·0	43·1
145·01	5·29	3·33	−3·49	5·90	39·1	47·0
146·18	4·33	3·16	−2·62	6·65	28·7	51·1
147·33	3·19	2·88	−1·68	7·17	18·5	54·2
148·49	1·93	2·45	−0·67	7·52	9·7	57·0
149·65	0·58	1·97	+0·36	7·62	4·2	58·2
150·2	−2·50	0·5	2·6	7·1	6·5	57
150·4	−4·82	−1·2	4·4	5·7	24·6	52
150·6	−6·81	−3·6	6·02	3·6	59·4	49
150·8	−8·17	−6·5	7·28	0·5	109	53
160·2	−9·58	−14·2	8·32	−8·1	294	135
160·4	−9·30	−18·1	8·18	−12·8	415	231
160·6	−8·42	−22·4	7·60	−17·9	573	378
160·8	−6·98	−26·9	6·36	−23·1	773	574
170·2	−3·20	−34·8	3·60	−33·0	1220	1100
170·4	−1·20	−38·0	2·18	−37·2	1445	1390
170·6	0·00	−40·2	1·36	−40·0	1620	1600
170·8	0·60	−41·5	0·82	−41·3	1720	1710

$\kappa a = 10$.

θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
4·58°			−2·085	−4·376		23·60
11·65	2·28	4·59	−2·420	−4·330	26·3	24·61
15·83	2·53	4·51	−2·750	−4·197	26·7	25·17
19·12	2·80	4·26	−3·066	−4·002	26·0	25·42
21·93	3·12	3·91	−3·369	−3·751	25·0	25·42
24·42	3·44	3·48	−3·656	−3·455	23·9	25·30
26·68	3·75	3·03	−3·924	−3·118	23·2	25·12
28·77	4·07	2·58	−4·168	−2·750	23·1	24·93
30·71	4·36	2·15	−4·386	−2·360	23·5	24·81

TABLE XXVI. (continued).

$\kappa a = 10$.						
θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
32°·55	4·64	1·72	−4·577	−1·945	24·5	24·73
34·29	4·87	1·27	−4·738	−1·520	25·3	24·76
35·95	5·06	0·86	−4·863	−1·085	26·3	24·83
37·53	5·20	0·47	−4·956	−0·644	27·3	24·98
39·06	5·27	0·10	−5·009	−0·200	27·8	25·13
41·95	5·24	−0·69	−4·993	0·678	27·9	25·39
44·66	5·02	−1·26	−4·825	1·530	26·8	25·62
47·22	4·59	−1·94	−4·498	2·328	24·8	25·65
49·66	3·96	−2·63	−4·029	3·049	22·6	25·53
51·99	3·21	−3·27	−3·430	3·680	21·0	25·31
54·22	2·45	−3·86	−2·720	4·200	20·9	25·04
56·38	1·64	−4·43	−1·926	4·604	22·3	24·90
58·46	0·82	−4·85	−1·068	4·878	24·2	24·87
60·48	0·00	−5·14	−0·181	5·000	26·4	25·03
62·43	−0·77	−5·30	+0·708	4·969	28·7	25·19
64·34	−1·48	−5·28	+1·577	4·790	30·1	25·43
66·19	−2·12	−5·01	2·390	4·470	29·6	25·69
68·00	−2·65	−4·55	3·133	4·006	27·7	25·87
69·77	−3·22	−3·92	3·777	3·425	25·7	26·00
71·50	−3·64	−3·15	4·300	2·731	23·2	25·95
73·19	−3·98	−2·23	4·698	1·949	20·8	25·87
74·85	−4·22	−1·25	4·943	1·101	19·4	25·64
76·48	−4·39	−0·23	5·044	0·216	19·3	25·49
79·68	−4·45	+1·76	4·786	−1·569	22·9	25·37
82·74	−4·21	3·41	3·950	−3·166	29·6	25·62
85·72	−3·38	4·60	2·633	−4·385	32·6	26·16
88·62	−2·04	5·11	0·998	−5·074	30·3	26·75
91·45	−0·59	4·87	−0·767	−5·139	24·1	27·00
94·21	+1·13	4·03	−2·457	−4·560	17·5	26·83
96·91	2·83	2·75	−3·861	−3·414	15·6	26·58
99·56	4·31	1·24	−4·809	−1·824	20·1	26·46
102·15	5·37	−0·30	−5·184	0·000	28·9	26·87
104·70	5·77	−1·62	−4·919	1·840	35·9	27·59
107·20	5·44	−2·62	−4·050	3·455	36·5	28·34
109·67	4·42	−3·25	−2·660	4·688	30·1	29·06
112·10	2·74	−3·39	−0·919	5·336	19·0	29·31
114·49	0·63	−3·23	+0·972	5·343	10·9	29·49
116·85	−1·65	−2·81	2·761	4·682	10·6	29·54
119·18	−3·82	−2·18	4·245	3·418	19·3	29·70

TABLE XXVI. (continued).

$\kappa a = 10.$

θ.	Y_1.	Y_2.	Z_1.	Z_2.	$Y_1^2+Y_2^2$.	$Z_1^2+Z_2^2$.
121·49°	− 5·55	− 1·31	5·243	+ 1·61	32·5	30·1
123·77	− 6·60	− 0·74	5·585	− 0·31	44·1	31·3
124·90	− 6·83	− 0·50	5·511	− 1·30	46·9	32·1
126·03	− 6·84	− 0·18	5·240	− 2·29	47·1	32·7
127·15	− 6·64	+ 0·07	4·795	− 3·23	44·1	33·4
128·26	− 6·20	+ 0·26	4·179	− 4·08	38·5	34·1
129·37	− 5·54	+ 0·48	3·462	− 4·82	30·9	35·2
130·47	− 4·67	+ 0·70	2·601	− 5·42	22·3	36·2
131·57	− 3·63	+ 0·91	1·662	− 5·84	14·0	36·9
132·66	− 2·47	+ 1·03	0·664	− 6·09	7·2	37·5
133·75	− 1·20	+ 1·33	− 0·358	− 6·18	3·2	38·2
134·84	+ 0·03	1·49	− 1·375	− 6·06	2·2	38·6
135·92	1·42	1·76	− 2·372	− 5·76	5·1	38·8
137·00	2·74	1·99	− 3·320	− 5·26	11·5	38·7
138·07	3·98	2·24	− 4·178	− 4·57	20·9	38·4
139·14	5·09	2·45	− 4·912	− 3·70	31·9	37·8
140·21	6·02	2·77	− 5·501	− 2·69	44·5	37·5
141·27	6·79	3·10	− 5·926	− 1·56	55·7	37·6
142·33	7·35	3·42	− 6·173	− 0·32	65·7	38·2
143·38	7·73	3·78	− 6·236	+ 1·00	74·0	39·9
144·43	7·87	4·01	− 6·112	2·32	78·0	43·7
145·48	7·72	4·29	− 5·816	3·63	78·0	47·0
146·53	7·33	4·41	− 5·358	4·88	73·2	52·5
147·57	6·64	4·46	− 4·736	6·00	64·0	58·4
148·61	5·75	4·40	− 3·934	6·96	52·4	63·9
149·65	4·64	4·20	− 2·982	7·75	39·2	69·0
151	3·00	3·80	− 1·50	8·68	23·1	77·6
152	1·78	3·29	− 0·34	9·00	14·0	81·1
153	0·19	2·62	1·00	9·15	6·9	84·7
154	− 1·32	1·80	2·20	8·93	5·0	84·5
155	− 2·74	0·83	3·30	8·40	7·2	81·5
156	− 4·02	− 0·39	4·50	7·53	16·3	77·0
157	− 5·32	− 1·73	5·44	6·22	31·3	68·3
158	− 6·72	− 3·59	6·40	4·33	58·1	59·7
159	− 7·78	− 5·50	7·27	2·60	90·8	59·7
162	− 10·62	− 12·45	9·17	− 4·80	268	107
164	− 10·96	− 18·2	9·5	− 11·0	451	210
166	− 10·58	− 23·5	9·1	− 17·3	664	380
168	− 9·00	− 29·5	7·9	− 24·8	951	680
172	− 4·76	− 41·0	4·7	− 38·0	1700	1470
174	− 2·33	− 45·6	3·0	− 43·9	2080	1940
176	− 0·23	− 49·3	1·6	− 48·4	2430	2350
178	+ 0·60	− 51·0	0·8	− 51·0	2600	2600

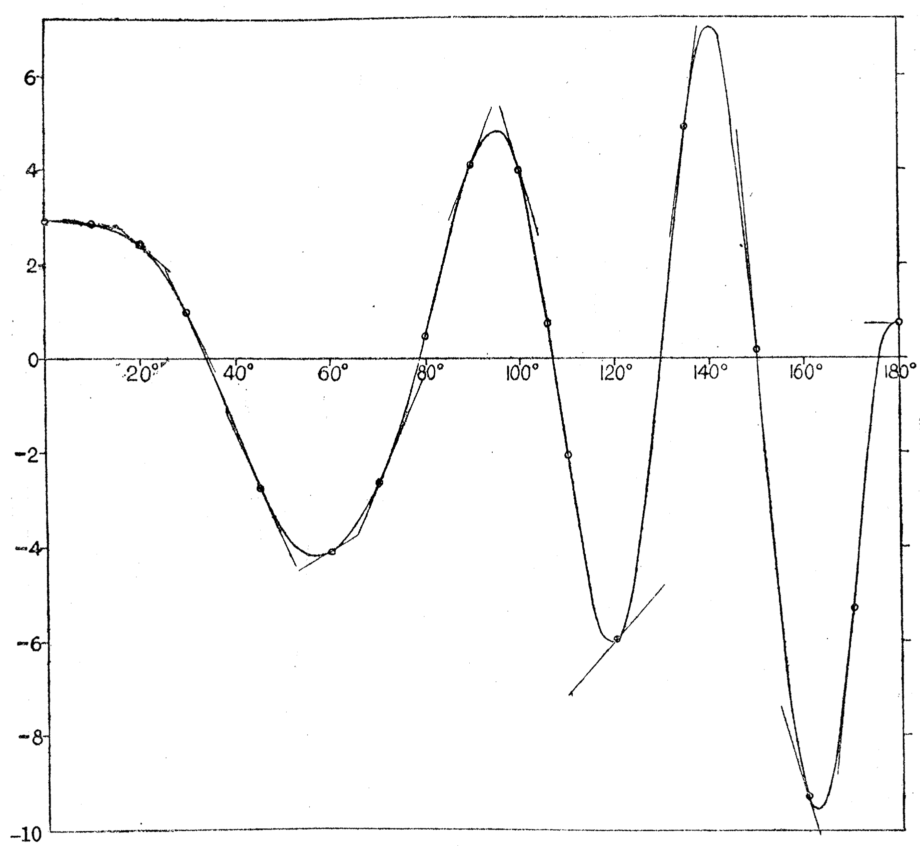

Fig. 17. Tables X., XVIII., XXVI. ($\kappa a = 9$) Y_1.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

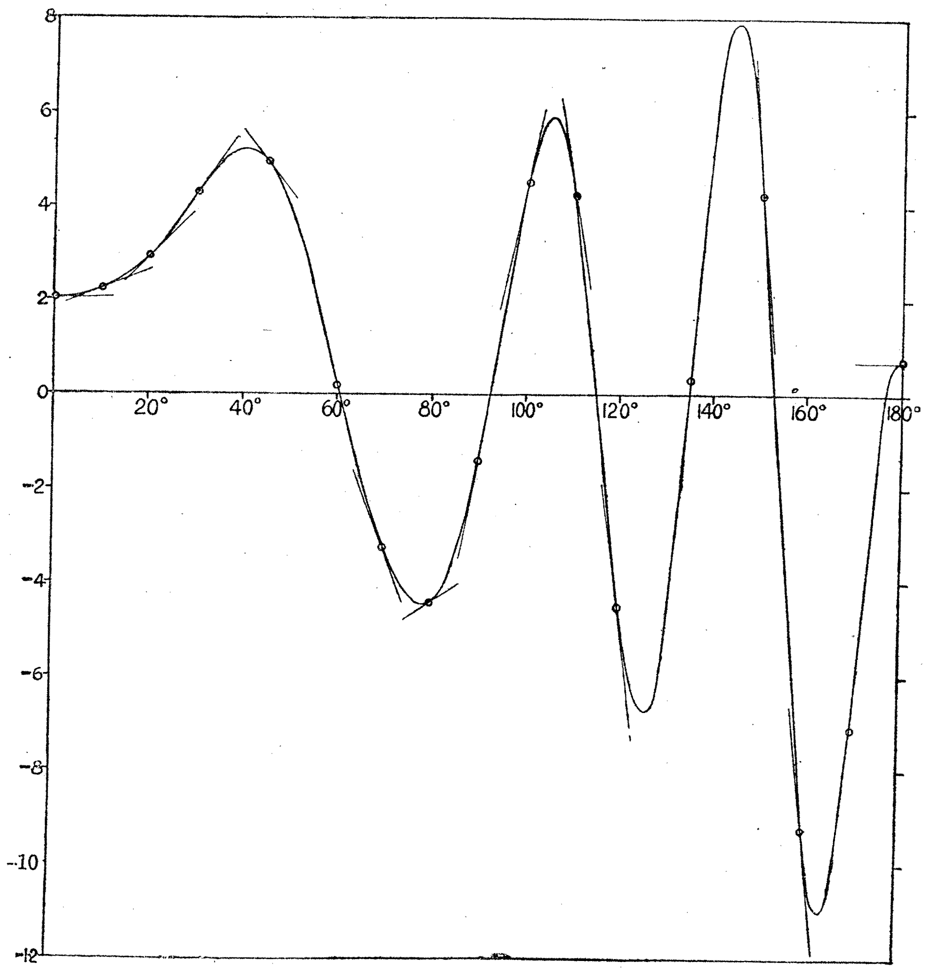

Fig. 18. Tables X., XVIII., XXVI. ($\kappa a = 10$) Y_1.

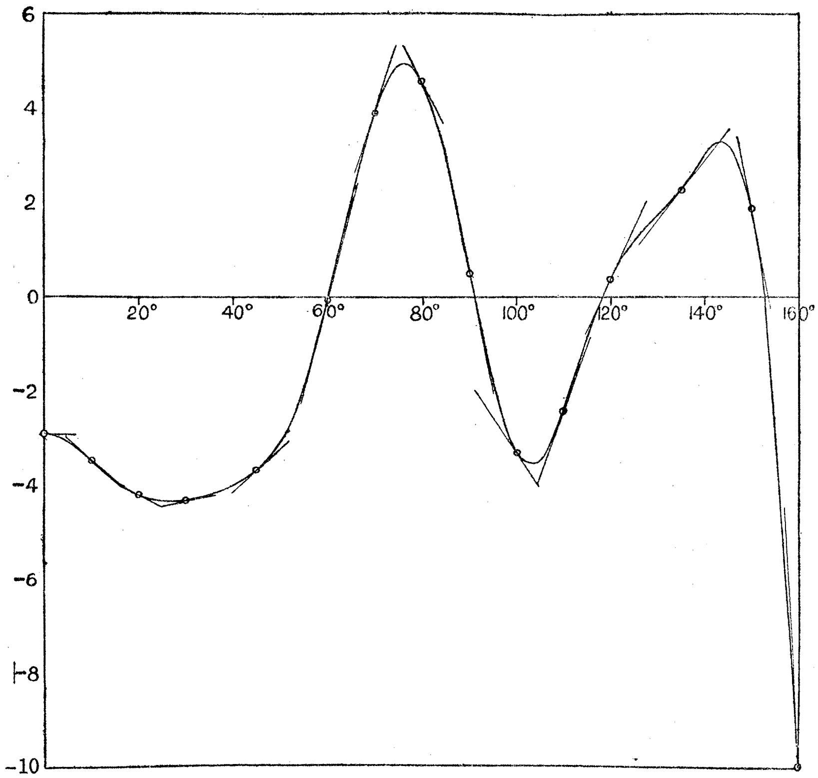

Fig. 19. Tables X., XVIII., XXVI. ($\kappa a = 9$) Y_2.

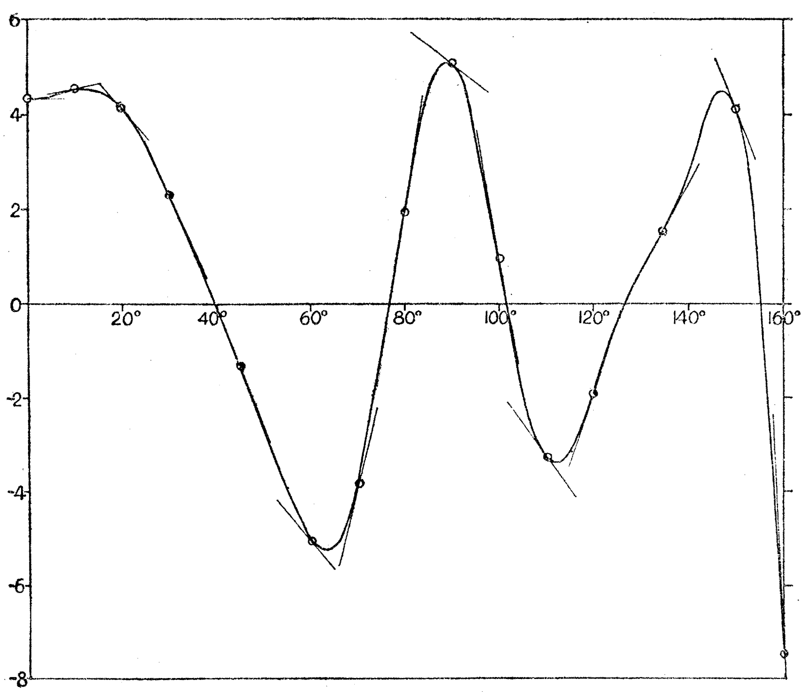

Fig. 20. Tables X., XVIII., XXVI ($\kappa a = 10$) Y_2.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

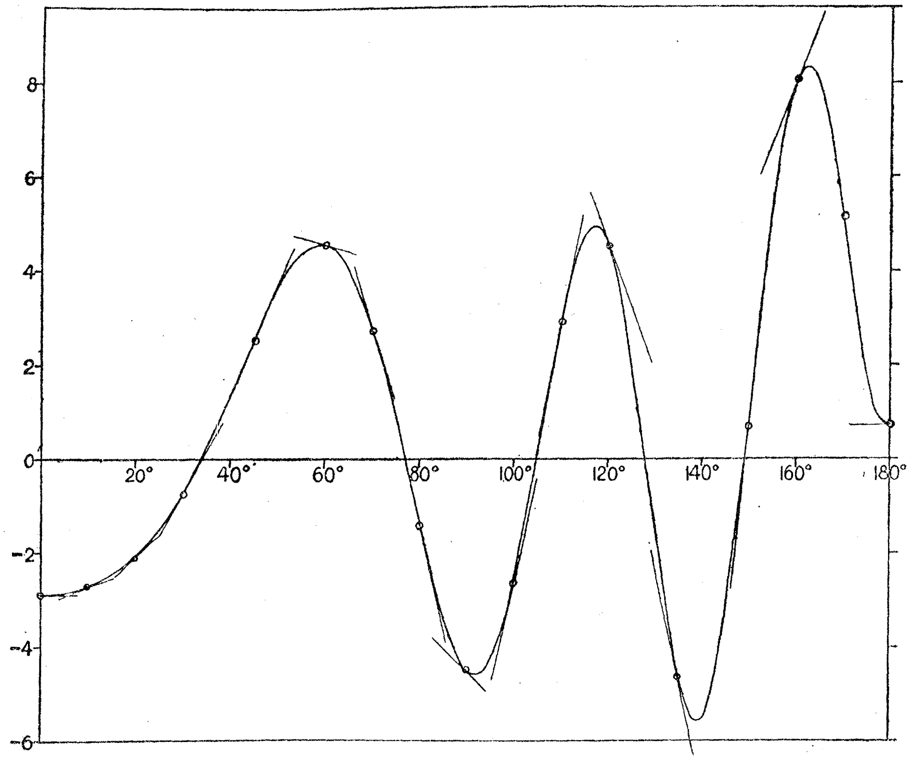

Fig. 21. Tables X., XVIII., XXVI. ($\kappa a = 9$) Z_1.

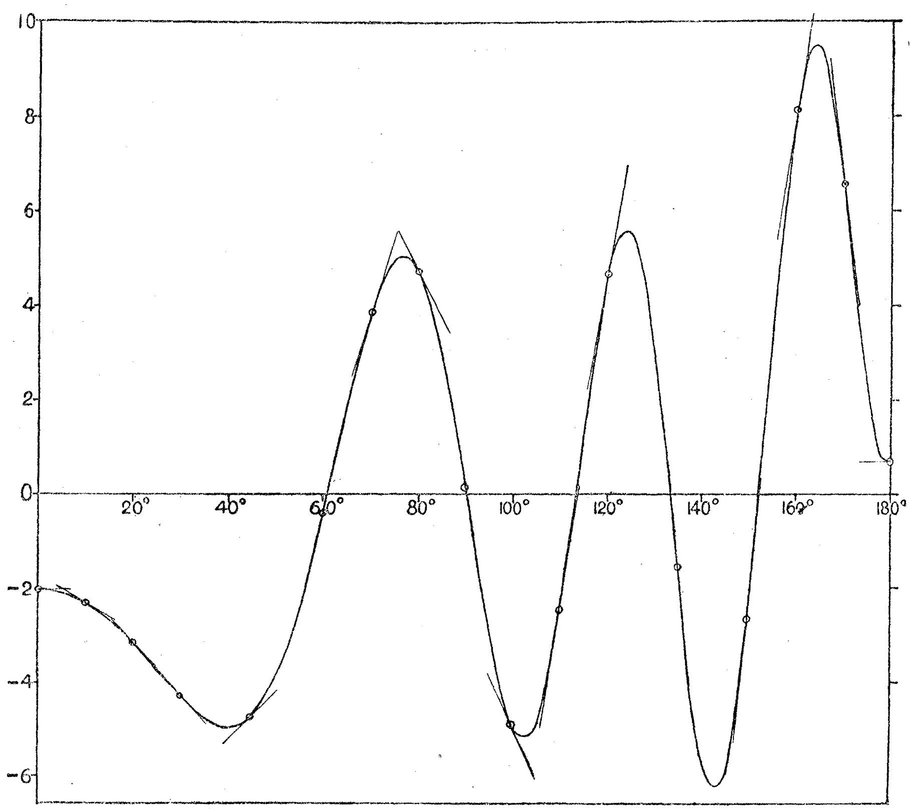

Fig. 22. Tables X., XVIII., XXVI. ($\kappa a = 10$) Z_1.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

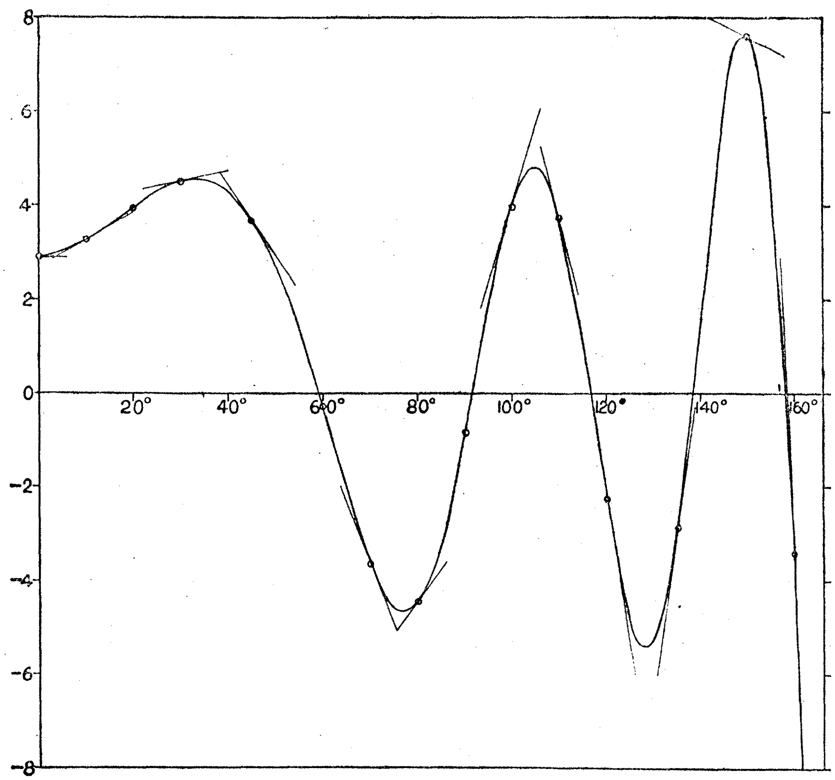

Fig. 23. Tables X., XVIII., XXVI. ($\kappa a = 9$) Z_2.

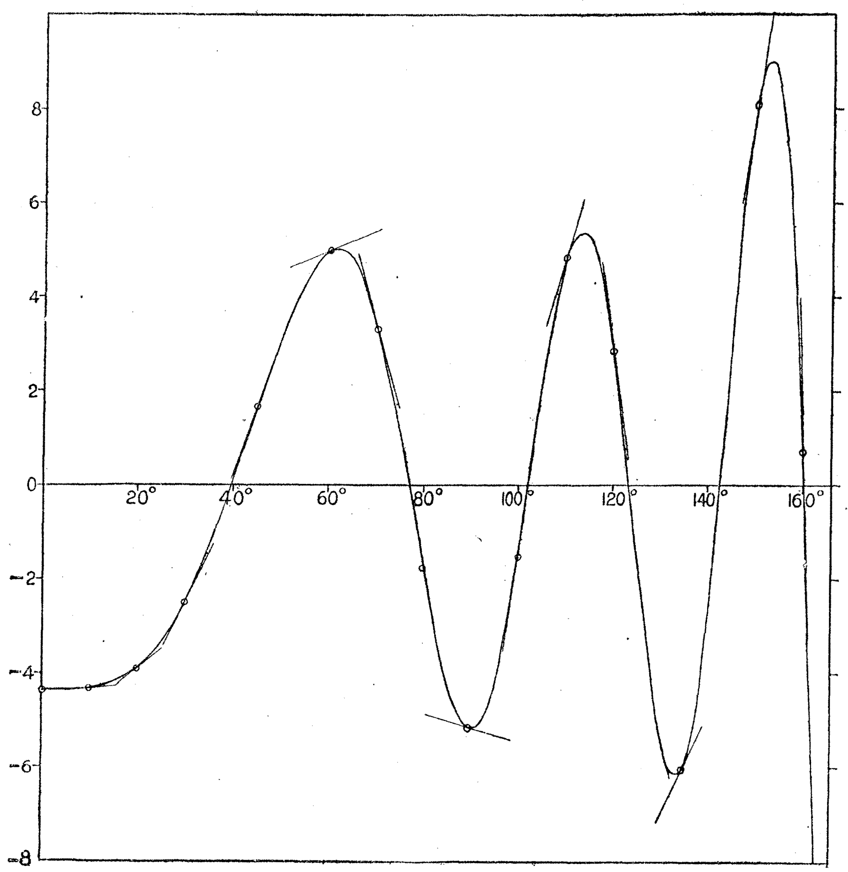

Fig. 24. Tables X., XVIII., XXVI. ($\kappa a = 10$) Z_2.

Figs. 25 *to* 28 *give the final graphs of* $Y_1^2+Y_2^2$, $Z_1^2+Z_2^2$, *the obtaining of which has been the object of the whole work.*

From $\theta = 0°$ to $120°$ the curve for $Z_1^2+Z_2^2$ is probably correct to 0·2 per cent.; from 120° to 150°, to 0·5 per cent.; from 150° to 180°, to 1 per cent. The curve for $Y_1^2+Y_2^2$ is probably correct to 1 per cent. throughout the range.

These curves are drawn to different scales, as $Z_1^2+Z_2^2$ is approximately constant up to $\theta = 120°$, while $Y_1^2+Y_2^2$ has oscillations of increasing amplitude. The curves for $Y_1^2+Y_2^2$ illustrate these oscillations up to the point where the last minimum occurs; after this the value of $Y_1^2+Y_2^2$ increases rapidly to a maximum at $\theta = 180°$. The curves for $Z_1^2+Z_2^2$ illustrate the behaviour of this function up to $\theta = 140°$, but do not show the last oscillation before the function begins to increase rapidly to its maximum at $\theta = 180°$. For this range data are given in the tables.

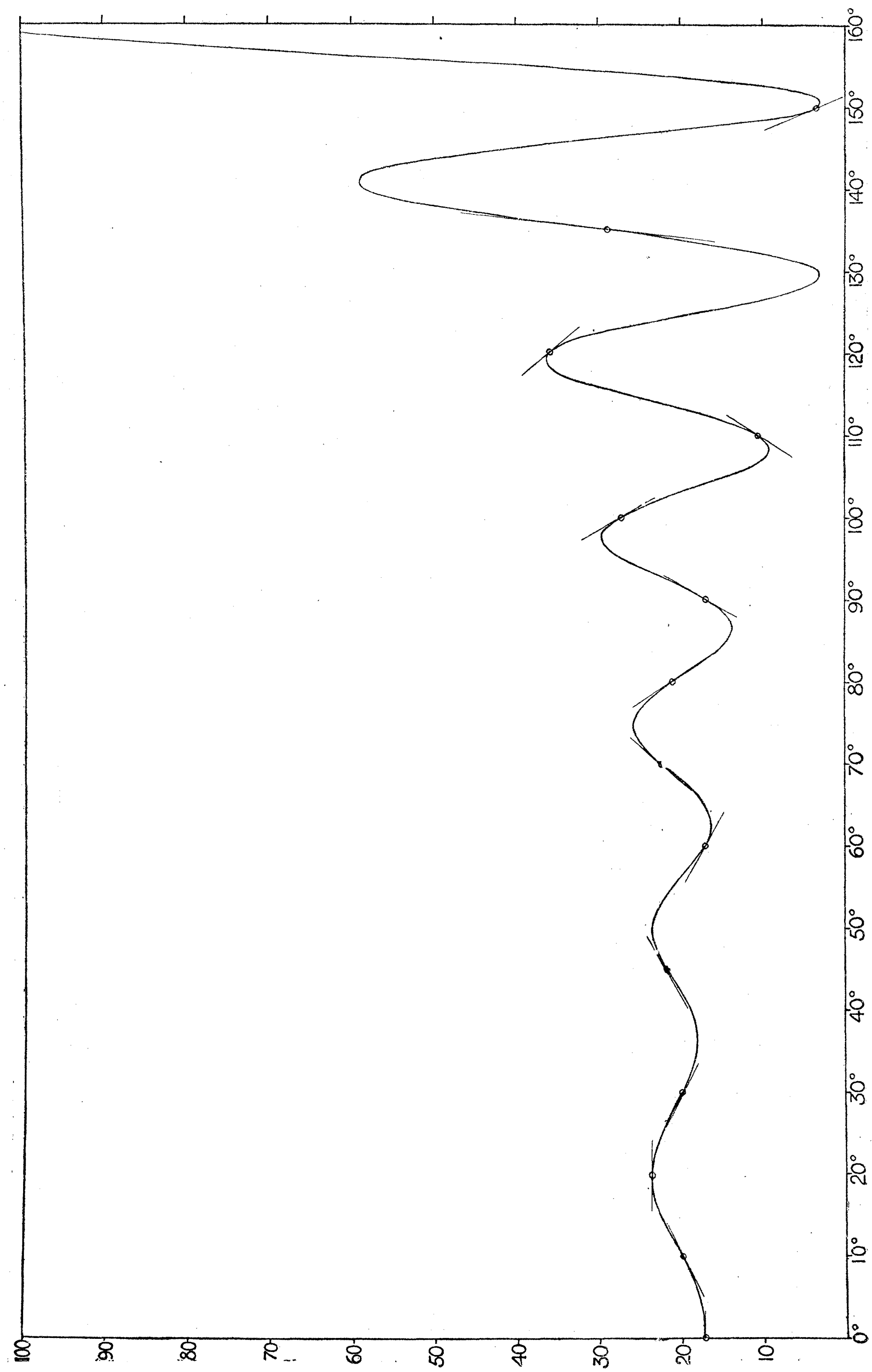

Fig. 25. Tables XI., XIX., XXVI. ($\kappa a = 9$) $Y_1^2 + Y_2^2$.

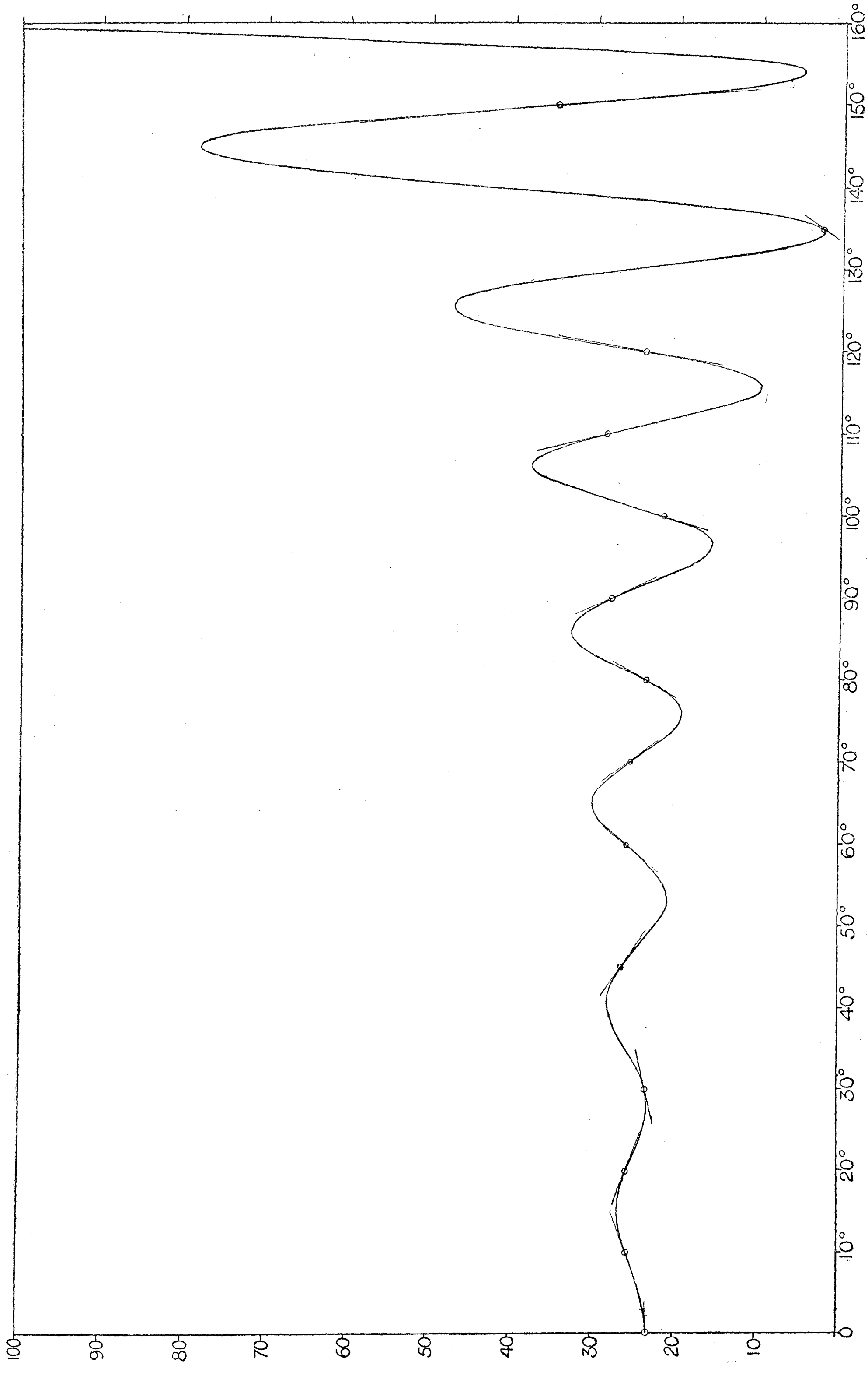

Fig. 26. Tables XI., XIX., XXVI. ($\kappa a = 10$) $Y_1^2 + Y_2^2$.

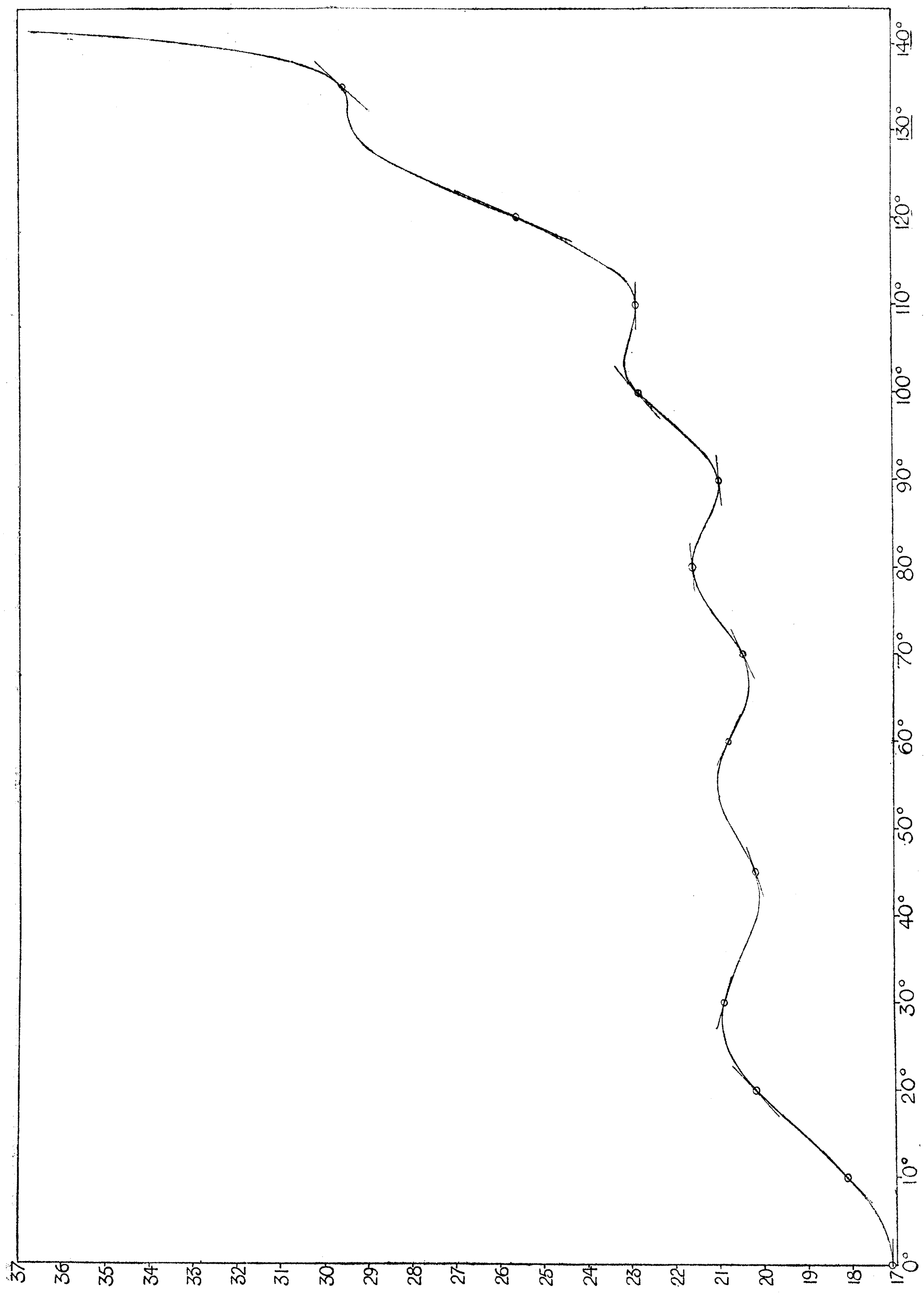

Fig. 27. Tables XI., XIX., XXVI. ($\kappa a = 9$) $Z_1^2 + Z_2^2$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

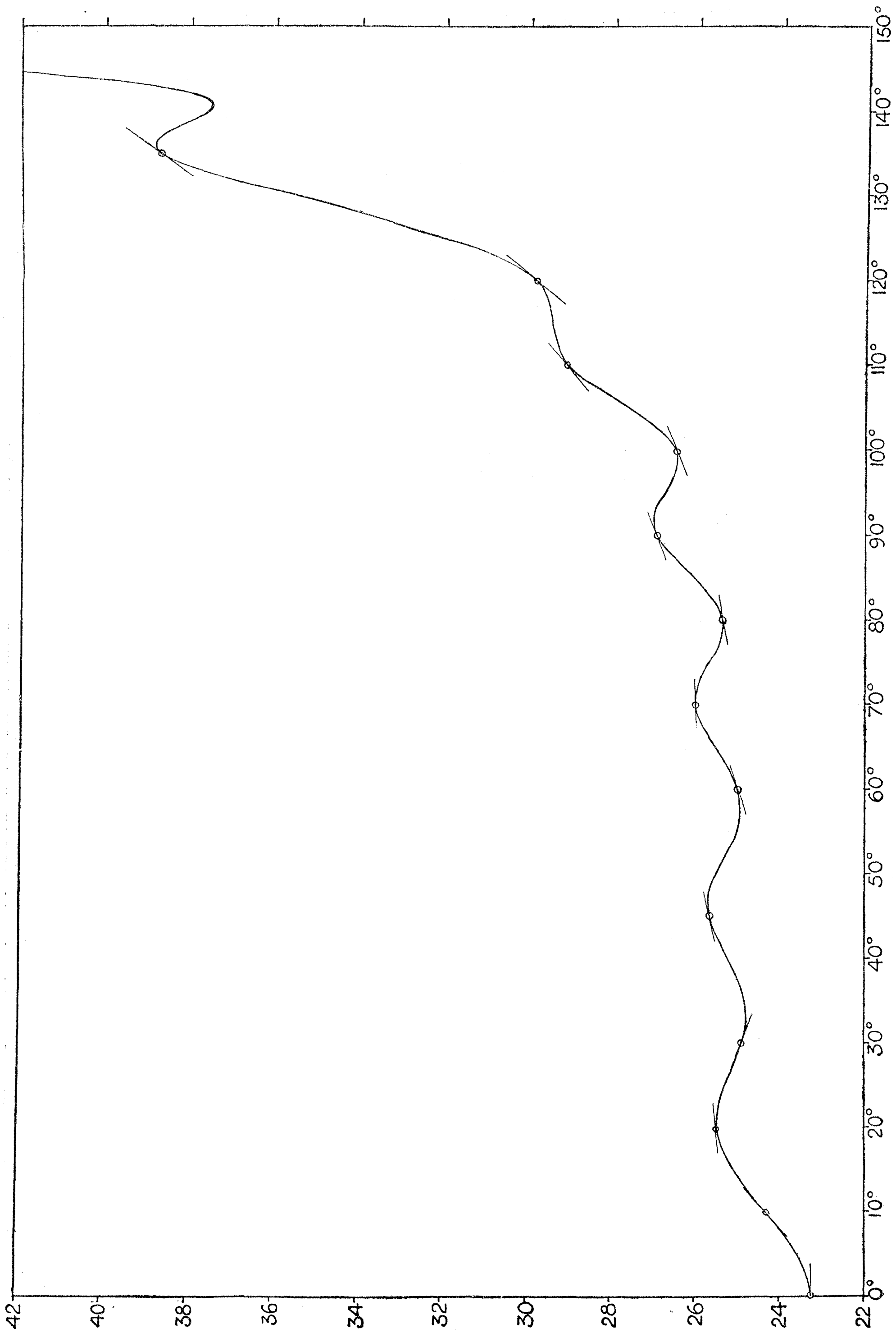

Fig. 28. Tables XI., XIX., XXVI. ($\kappa a = 10$) $Z_1^2 + Z_2^2$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES